INTRODUCTION TO THE INFINITESIMAL CALCULUS

H. S. CARSLAW

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NOTES FOR THE USE OF SCIENCE AND ENGINEERING STUDENTS

BY

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SECOND EDITION

LONGMANS, GREEN AND CO.
39 PATERNOSTER ROW, LONDON
NEW YORK, BOMBAY AND CALCUTTA
1912

PREFACE TO THE FIRST EDITION

THESE introductory chapters in the Infinitesimal Calculus were lithographed and issued to the students of the First Year in Science and Engineering of the University of Sydney at the beginning of last session. They form an outline of, and were meant to be used in conjunction with, the course on *The Elements of Analytical Geometry and the Infinitesimal Calculus*, which leads up to a term's work on Elementary Dynamics.

The standard text-books amply suffice for the detailed study of this subject in the second year, but the absence of any discussion of the elements and first principles suitable for the first year work, was found to be a serious hindrance to the work of the class. For such students a separate course on Analytical Geometry, without the aid of the Calculus, is not necessary, and the exclusion of the methods of the Calculus from the analytical study of the Conic Sections is quite opposed to the present unanimous opinion on the education of the engineer. It has been our object to present the fundamental ideas of the Calculus in a simple manner and to illustrate them by practical examples, and thus to enable these students to use its methods intelligently and readily in their Geometrical, Dynamical, and Physical work early in their University course. This little book is not meant to take the place of the standard treatises on the subject, and, for that reason, no attempt is made to do more than give the lines of the proof of some of the later theorems. As an introduction to these works, and as a special text-book for such a "short course" as is found necessary in the engineering schools of the Universities and in the Technical Colleges, it is hoped that it may be of some value.

In the preparation of these pages I have examined most of the standard treatises on the subject. To Nernst and Schönflies' *Lehrbuch der Differential- und Integral-Rechnung*, to Vivanti's *Complementi di Matematica ad uso dei Chemici e dei Naturalisti*, to Lamb's *Infinitesimal Calculus*, and to Gibson's *Elementary Treatise on the Calculus*, I am conscious of deep obligations. I should also add that from the two last-named books, and from those of Lodge, Mellor, and Murray, many of the examples have been obtained.

In conclusion, I desire to tender my thanks to my Colleagues in the University of Sydney, Mr. A. Newham and Mr. E. M. Moors, for assistance in reading the proof-sheets; to my students, Mr. D. R. Barry and Mr. R. J. Lyons, for the verification of the examples; also to my old teacher, Professor Jack of the University of Glasgow, and to Mr. D. K. Picken and Mr. R. J. T. Bell of the Mathematical Department of that University, by whom the final proofs have been revised.

H. S. CARSLAW.

THE UNIVERSITY OF SYDNEY,
June, 1905.

PREFACE TO THE SECOND EDITION

THE principal change in this edition will be found in the treatment of the exponential and logarithm. Six years ago few students began the study of the Calculus without having already completed a course in Algebra, including the Theory of Infinite Series. It is now realised that in making this demand the mathematical teacher was asking more than was necessary. The principles underlying the Calculus, in so far as they can be examined in such a course as this, offer little difficulty. No more than an elementary knowledge of Algebra and Trigonometry is required for their discussion; and a real grasp of the meaning of differentiation and integration can be obtained by very many to whom the subject of Infinite Series would appear extremely obscure.

These altered conditions have allowed me to place the older proofs of the theorems regarding the differentiation of e^x and $\log x$ in an Appendix, and I have introduced into the text one of the simpler methods, in which use is made of the Logarithm Tables. In this discussion I have followed the lines laid down by Love in his *Elements of the Differential and Integral Calculus.* However it seemed worth while to carry the numerical work a little further, with the help of 8-Figure and 15-Figure Tables. The student is apt to imagine that 4-Figure and even 7-Figure Tables give a more accurate result than they frequently afford.

The other changes that need be mentioned are the addition of a section on Repeated Differentiation, and one on Fluid Pressure. A number of easy examples and of graphical illustrations have also been inserted.

H. S. CARSLAW.

SYDNEY, *December*, 1911.

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CHAPTER I

THE ANALYTICAL GEOMETRY OF THE STRAIGHT LINE

§ 1. Cartesian Co-ordinates.

The position of a point on a plane may be fixed in different ways. In particular it is determined if its distances from two fixed perpendicular lines in the plane are known, the usual con-

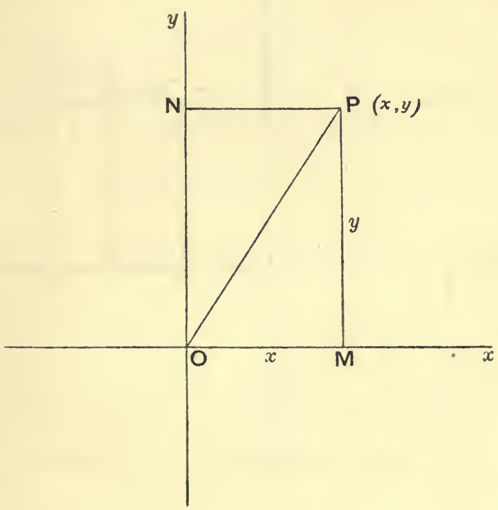

Fig. 1.

ventions with regard to sign being adopted. These two lines Ox and Oy are called the axes of x and y; and the lengths OM and ON, which the perpendiculars from the point P cut off from the axes, are called the co-ordinates of the point P and denoted by x and y. OM and ON are taken positive or negative according as they are measured along Ox and Oy, or in the opposite directions. OM is called the "*abscissa*" of P and MP is called the "*ordinate*" of P.

Ex. 1. Mark on a piece of squared paper the position of the points $(\pm 2, \pm 3)$.

2. Prove that the distance between the points (2, 3), and (−2, −3) is $2\sqrt{13}$.

3. Prove that the distance d between the points (x_1, y_1), (x_2, y_2) is given by

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

4. Prove that the co-ordinates of any point (x, y) upon the circle whose centre is at the point (a, b) and whose radius is c satisfy the equation

$$(x - a)^2 + (y - b)^2 = c^2.$$

§ 2. **The Co-ordinates of a Point dividing the Line joining two given Points in a given Ratio $l : m$.**

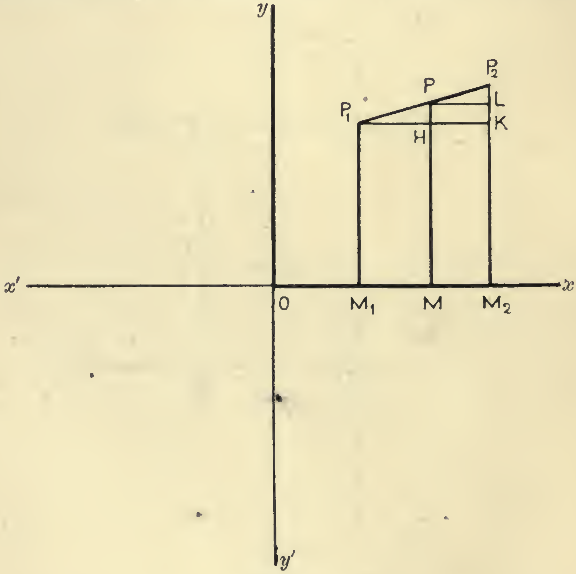

FIG. 2.

Let P_1 and P_2 be the two given points (x_1, y_1), (x_2, y_2); and let $P(x, y)$ divide P_1P_2 in the ratio $l : m$ (see Fig. 2).

Draw P_1M_1, PM and P_2M_2 perpendicular to Ox.

Also draw P_1HK and PL parallel to Ox.

Let these lines meet PM and P_2M_2 in H, K and L.

Since $$\frac{P_1H}{PL}=\frac{P_1P}{PP_2}=\frac{l}{m},$$

we have $$\frac{x-x_1}{x_2-x}=\frac{l}{m}.$$

$$\therefore\ x(l+m)=lx_2+mx_1.$$

$$\therefore\ \mathbf{x}=\frac{\mathbf{lx_2+mx_1}}{\mathbf{l+m}}.$$

Similarly $$\mathbf{y}=\frac{\mathbf{ly_2+my_1}}{\mathbf{l+m}}.$$

These are the co-ordinates of the internal point of section. Those of the external point may be found in the same way to be

$$x=\frac{lx_2-mx_1}{l-m}$$

and $$y=\frac{ly_2-my_1}{l-m}.$$

As a particular case of this theorem **the co-ordinates of the middle point of the line joining ($\mathbf{x_1}$, $\mathbf{y_1}$) and ($\mathbf{x_2}$, $\mathbf{y_2}$) are**

$$\frac{\mathbf{x_1+x_2}}{2}\quad\textbf{and}\quad\frac{\mathbf{y_1+y_2}}{2}.$$

Ex. 1. Prove that the co-ordinates of the middle point of the line which cuts off unit length from Ox and Oy are $\frac{1}{2}$ and $\frac{1}{2}$.

2. Find the co-ordinates of the points of trisection of this line, and also of the points which divide it externally in the ratio 1 : 2.

3. Prove that the C.G. of the triangle whose angular points are (2, 1), (4, 3), (2, 5) is the point $\left(\frac{8}{3}, 3\right)$; and give the general theorem.

§ 3. The Equation of the First Degree represents a Straight Line.

If the point P moves along a curve, the co-ordinates of the point are not independent of each other. In mathematical language "y is a function of x"; and we speak of $y=f(x)$ as *the equation of the curve,* meaning that all the points whose co-ordinates satisfy this equation lie upon the curve, and that the co-ordinates of all points upon the curve satisfy the equation.

For example, the equation of the circle with its centre at the point (a, b) and radius c is $(x-a)^2+(y-b)^2=c^2$. (Cf. § 1, Ex. 4.)

The same ideas are employed in Solid Geometry: the surface of a solid is represented by an equation satisfied by the co-ordinates of the points lying upon it; and straight lines and curves are given by simultaneous equations. The geometrical properties of curves and surfaces may often be obtained by discussing their equations. This branch of mathematics is called Analytical Geometry.

The simplest equation in the two variables x, y is that of the first degree

$$ax+by+c=0,$$

a, b and c being constants.

For example, take the equation

$$x+2y=4.$$

By assigning any value to x and solving the equation for y, we obtain, as in the accompanying table, the co-ordinates of any number of points upon the locus. Plotting these points upon squared paper in the usual way, we see that they all lie upon a straight line; and, so far as our measurements could be relied upon, we could verify that the co-ordinates of any point upon this line would satisfy the equation.

x	y
−3	3·5
−2	3
−1	2·5
0	2
1	1·5
2	1
3	·5

We proceed to prove that this is true in general: in other words, that *all the points whose co-ordinates satisfy the equation*

$$ax+by+c=0$$

lie upon one and the same straight line, and that the co-ordinates of all points upon this straight line satisfy the equation.

(i.) We consider first of all the equation

$$y=mx, \tag{1}$$

m being any real number.

Let P be any point whose co-ordinates x, y satisfy this equation. (Cf. Fig. 1.)

Draw PM perpendicular to the axis of x and join OP.

Let OP make an angle θ with the positive direction of the axis of x.

Then $$\tan\theta=\frac{\text{MP}}{\text{OM}}=\frac{y}{x}=m.$$

Therefore the point P lies upon the straight line through the origin which makes an angle whose tangent is m with the positive direction of the axis of x.

As the number m is a given number, this line is a definite straight line.

Now let us take any point upon this straight line and draw the perpendicular from that point to the axis of x.

It will be seen that the co-ordinates of the point satisfy the given equation.

It follows that every point whose co-ordinates satisfy the equation

$$y = mx$$

lies upon a certain straight line through the origin, and that the co-ordinates of every point upon this line satisfy the equation.

If $m > 0$, the line will be in the first and third quadrants.

If $m = 0$, the line is the axis of x, and if $m = \infty$, the line is the axis of y.

If $m < 0$, the line is in the second and fourth quadrants.

(ii.) We next consider the equation

$$y = mx + n, \tag{2}$$

where m and n are any real numbers.

For any value of x there is one and only one value of y. This value is greater by n than that for the corresponding point on the straight line given by $y = mx$.

Hence, to obtain all the points whose co-ordinates satisfy equation (2), *we have only to lengthen the ordinates of all the points on the straight line*

$$y = mx$$

by an amount n.

In other words, we have only to move this whole line parallel to itself through the distance n in the direction of the axis of y.

Or, more simply, we have to draw the parallel through the point $(0, n)$ to the line $y = mx$.

If $n > 0$, the point lies on the positive portion of the axis of y; if $n < 0$, it lies on the negative portion.

The co-ordinates of all points upon this line satisfy equation (2); and the co-ordinates of all points which do not lie upon this line do not satisfy equation (2).

(iii.) Finally we consider the equation

$$ax + by + c = 0. \tag{3}$$

If $b = 0$, x remains constant and the equation represents a line parallel to the axis of y.

If $b \neq 0$, we can write the equation in the form

$$y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right).$$

On putting $\quad m = -\frac{a}{b} \quad$ and $\quad n = -\frac{c}{b},$

this becomes $\quad y = mx + n,$

the form we have discussed in (ii.).

It follows that the equation

$$ax + by + c = 0$$

always represents a straight line.

For this reason the equation of the first degree is usually called a *linear* equation.

In the above discussion we started with the equation, and found that the locus which it represents is a straight line.

If a straight line is given, we can easily show that the co-ordinates of any point upon it satisfy a linear equation, which can always be obtained.

If the line is parallel to the axis of x, it is clear that the ordinates of all points upon it are the same. Its equation is thus $y = \text{const.}$

If it is parallel to the axis of y, its equation is $x = \text{const.}$

If it makes an angle θ with the positive direction of the axis of x, and cuts off an intercept n from the axis of y, we have seen that its equation is $y = mx + n$, where $\tan\theta = m$.

We thus speak of the *equation of the given straight line*, and we know that it is always of the form

$$ax + by + c = 0,$$

where a, b and c are constants which arise in specifying the line.

§ 4. Drawing Straight Lines from their Equations.

In the last article we have shown that the equation of the first degree represents a straight line. In plotting the locus given by such an equation, we do not now need to obtain a table of values of x and y, as we did above in the example $x + 2y = 4$. Two points fix a straight line. Therefore we have only to find two points whose co-ordinates satisfy the equation. The most convenient points are those where the line cuts the axes, and these are found by putting $x = 0$ and $y = 0$, respectively, in the equation.

Ex. 1. Draw the lines (i.) $x=0, \quad x=1, \quad x=-1$;

(ii.) $y=0, \quad y=2, \quad y=-2$;

(iii.) $x+y=0, \quad x+y=1$;

(iv.) $y=2x, \quad y=2x+3$;

(v.) $\frac{x}{4}+\frac{y}{3}=1, \quad \frac{x}{4}-\frac{y}{3}=1.$

2. Determine whether the point (2, 3) is on the line

$$4x+3y=15.$$

3. What is the condition that the point (a, b) should lie upon the line

$$ax+by=2ab?$$

§ 5. **The Gradient of a Line.**

When we speak of "the gradient" of a road being 1 in 200 we usually mean that the ascent is 1 foot vertical for 200 feet horizontal. This might also be called the slope of the road. The same expression is used with regard to the straight line. The "gradient" or the "slope" of a straight line is its rise per unit horizontal distance; or the ratio of the increase in y to the increase in x as we move along the line. This is evidently the same at all points of the straight line, and is equal to the tangent of the angle the line makes with the axis of x measured in the positive direction.

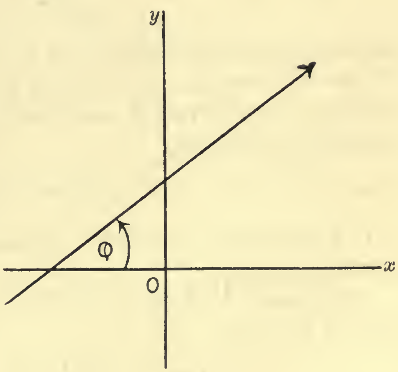

FIG. 3.

To save ambiguity it is well to fix upon the angle to be chosen, and in these pages it will be convenient to consider the line as always drawn upward in the direction $y>0$ (Fig. 3), and thus to restrict the angle ϕ to lie between 0° and 180°. It is convenient to speak of the line as drawn in the positive direction in such a case.

When $0<\phi<\frac{\pi}{2}$ the gradient is positive.

When $\frac{\pi}{2}<\phi<\pi$ the gradient is negative.

Ex. Write down the values of ϕ for the lines in § 4 (i.).

§ 6. **Different Forms of the Equation of the Straight Line.** In the preceding articles we have shown that the equation

$$ax + by + c = 0$$

represents a straight line, and we have seen how the line may be drawn when its equation is given. We shall now show *how to obtain the equation of the line when two points upon it are given.*

Let (x_1, y_1), (x_2, y_2) be the two given points. Let (x, y) be the co-ordinates of any point upon the line. Then it is clear (cf. Fig. 2) that

$\dfrac{y - y_1}{x - x_1}$ is equal to the gradient of the line,

and that $\dfrac{y_2 - y_1}{x_2 - x_1}$ is also equal to the gradient of the line.

Thus we have the equation

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

between the co-ordinates (x, y) of the representative point and the co-ordinates $(x_1, y_1)(x_2, y_2)$ of the fixed points. This is the equation of the straight line through these points. It is more conveniently written

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}. \qquad \text{(A)}$$

It follows from the above argument, or can be proved independently, that

The equation of the line through (x_1, y_1), making an angle ϕ with the axis of x, is

$$\frac{y - y_1}{x - x_1} = \tan\phi; \qquad \text{(B)}$$

and that

The equation of the line which cuts off a length c from the axis of y, and is inclined at an angle whose tangent is m to the axis of x, is

$$y = mx + c; \qquad \text{(C)}$$

and that

The equation of the line which cuts off intercepts a and b from the axis of x and y is

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad \text{(D)}$$

Ex. 1. Write down the equations of the lines through the following pairs of points: (1, 1), (1, -1); (1, 2), (-1, -2); (3, 4), (5, 6); (a, b), (a, $-b$).

2. Find the equations of the lines through the point (3, 4) with gradient ± 5, and draw the lines.

3. The lines $y=x$ and $y=2x$ form two adjacent sides of a parallelogram, the opposite angular point being (4, 5). Find the equations of the other two sides; and of the diagonals.

4. Write down the equations of the lines making angles 30°, 45°, 60°, 120°, 135°, and 150° with the axis of x, which cut this axis at unit distance from the origin in the negative direction.

§ 7. The "Perpendicular" Form of the Equation of the Straight Line.

A straight line is determined when the length of the perpendicular upon it from the origin, and the direction of this perpendicular are given.

Let ON be the perpendicular, p, upon the line.

Let the angle between ON and Ox be α, this angle lying between 0 and 2π (cf. Fig. 4).

Then N is the point ($p\cos\alpha$, $p\sin\alpha$).

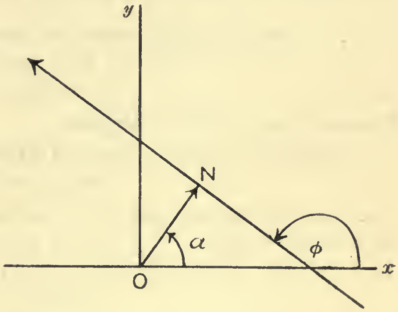

Fig. 4.

Using the form (B) of § 6, the equation of the line becomes

$$\frac{y-p\sin\alpha}{x-p\cos\alpha}=\tan\phi=\tan\left(\alpha+\frac{\pi}{2}\right)=-\frac{\cos\alpha}{\sin\alpha}.$$

This reduces to

$$x\cos\alpha+y\sin\alpha=p. \qquad \text{(E)}$$

N.B.—The quantity p is to be taken always positive, and the angle α is the angle between $\overrightarrow{\mathrm{O}x}$ and $\overrightarrow{\mathrm{ON}}$.

§ 8. The Point of Intersection of Two Straight Lines.

Since the point of intersection of the two lines

$$ax+by+c=0,$$
$$a'x+b'y+c'=0$$

lies on both lines, its co-ordinates x, y satisfy both equations.

Solving the equations we have

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{1}{ab'-a'b}.$$

It is clear that if $$ab'-a'b=0,$$

and neither of the other two denominators vanish, the co-ordinates x, y are infinite, and the lines are parallel.

If in addition $$ca'-c'a=0,$$

we have $$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'},$$

and the third denominator $bc'-b'c$ also vanishes.

In this case the two equations are not independent, and they really represent the same straight line.

Ex. 1. Find the co-ordinates of the point of intersection of the lines

$$2x+\ y=4,$$
$$x+2y=6.$$

Illustrate your result by a diagram.

2. Find the equations of the lines through (2, 3) parallel to

$$3x\pm 4y=5.$$

3. Find the co-ordinates of the angular points of the triangle whose sides are given by

$$x+\ y=2, \quad (1)$$
$$3x-2y=1, \quad (2)$$
$$4x+3y=24. \quad (3)$$

Also find the equations of the medians of this triangle and the co-ordinates of its C.G.

§ 9. The Angle between Two Straight Lines whose Equations are given.

When one of the lines is parallel to the axis of y, the angle between them can be readily found.

In all the other cases the equations can be reduced to the forms

$$y=mx+c, \quad (1)$$
$$y=m'x+c'. \quad (2)$$

Also the angle between these lines is the same as the angle between the lines

$$y=mx \quad (3)$$

and $$y=m'x, \quad (4)$$

which pass through the origin and are parallel to the given lines.

Let OQ, OQ′ (cf. Fig. 5) be the positive directions of the lines (3) and (4), with gradients m and m', respectively.

Let $$\angle xOQ = \phi \qquad (0 < \phi < \pi)$$

and $$\angle xOQ' = \phi'. \qquad (0 < \phi' < \pi)$$

Then $$\tan \phi = m$$

and $$\tan \phi' = m'.$$

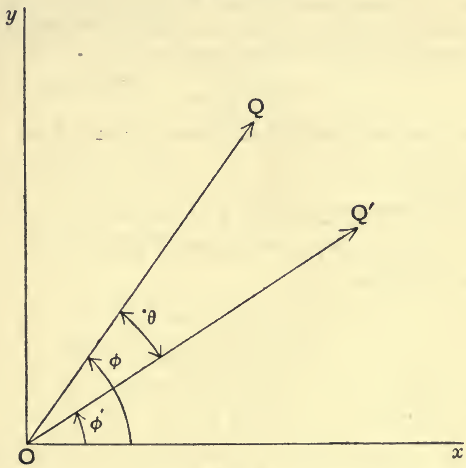

FIG. 5.

Let ϕ be the larger of these two angles, and let θ be the angle between the lines.

Then $$\theta = \phi - \phi'.$$

Therefore $$\tan \theta = \tan(\phi - \phi')$$

$$= \frac{\tan \phi - \tan \phi'}{1 + \tan \phi \tan \phi'}$$

$$= \frac{m - m'}{1 + mm'}.$$

Since there is only one angle θ less than two right angles, which satisfies this equation, it can be solved without ambiguity.

If $\dfrac{m - m'}{1 + mm'}$ is positive, the angle θ is acute.

If $\dfrac{m - m'}{1 + mm'}$ is negative, it is obtuse.

We have thus proved that the absolute *value of*

$$\frac{m-m'}{1+mm'}$$

is equal to the tangent of the acute *angle between the lines*

$$y=mx+c,$$
$$y=m'x+c'.$$

In practice it is unnecessary either to draw the lines, or to consider which has the greater slope. Taking the lines in any order, we need only calculate the absolute value of the expression

$$\frac{m-m'}{1+mm'}.$$

The acute angle between the lines can then be written down.

It follows that

(i.) *The lines are parallel, if* $m=m'$;

(ii.) *The lines are perpendicular, if* $mm'+1=0$.

When the equations are

$$ax+by+c=0,$$

and $$a'x+b'y+c'=0,$$

the *absolute* value of $$\frac{ab'-a'b}{aa'+bb'}$$

is equal to the tangent of the *acute* angle between the lines.

It follows that

(i.) *The lines are parallel, if* $\frac{a}{a'}=\frac{b}{b'}$;

(ii.) *The lines are perpendicular, if* $aa'+bb'=0$.

Ex. 1. Write down the equation of the straight line through (1, 2) perpendicular to $x-y=0$.

2. Find the angles between the lines

$$\left.\begin{array}{r}x-2y+1=0\\ x+3y+2=0\end{array}\right\}$$

and

$$\left.\begin{array}{r}4x+3y=12\\ 3x+4y=12\end{array}\right\}$$

and draw the lines.

3. Write down the equation of the straight line through (a, b) perpendicular to $bx-ay=a^2+b^2$.

4. Write down the equation of the line bisecting the line joining (1, 2), (3, 4) at right angles, and the equations of the perpendiculars upon both lines from the origin.

5. *Prove that* $l(x-a)+m(y-b)=0$ *is the equation of the line through* (a, b) *parallel to* $lx+my=0$; *and that* $m(x-a)-l(y-b)=0$ *is the equation of the line through* (a, b) *perpendicular to* $lx+my=0$.

6. Write down the equations of the lines through the C.G. of the triangle whose angular points are at $(4, -5)$, $(5, -6)$, $(3, 1)$ parallel and perpendicular to the sides.

§ 10. The Length of the Perpendicular from a Point (x_0, y_0) upon a Straight Line whose Equation is given.*

(i.) If the equation of the straight line is given in the "perpendicular" form

$$x \cos \alpha + y \sin \alpha = p, \quad (1)$$

the line through $P(x_0, y_0)$ parallel to it is given by

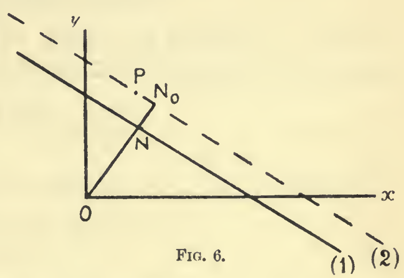

FIG. 6.

$$(x - x_0) \cos \alpha + (y - y_0) \sin \alpha = 0, \quad (2)$$

that is, by $\quad x \cos \alpha + y \sin \alpha = x_0 \cos \alpha + y_0 \sin \alpha.$

But if p_0 is the perpendicular ON_0 from O upon the line (2), and if N, N_0 are on the same side of O, the equation of PN_0 may be written $\quad x \cos \alpha + y \sin \alpha = p_0.$

Since (x_0, y_0) lies upon PN_0, we have

$$x_0 \cos \alpha + y_0 \sin \alpha = p_0.$$

Also the perpendicular from $P(x_0, y_0)$ upon the line (1) is

$$ON_0 - ON, \quad \text{(cf. Fig. 6)}$$

i.e. $\quad p_0 - p,$

i.e. $\quad x_0 \cos \alpha + y_0 \sin \alpha - p.$

In the case when N_0 lies between O and N, we have to take

$$p - p_0;$$

and when N, N_0 lie on opposite sides of O, ON_0 makes an angle $(\alpha + \pi)$ with Ox, and we have to take

$$p + p_0.$$

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* This section, and the examples in which it is required, may be omitted by those who only require such a knowledge of analytical geometry as is necessary for the pages of this book referring to the Calculus.

In both these cases the length of the perpendicular is given by

$$-x_0 \cos \alpha - y_0 \sin \alpha + p.$$

(ii.) If the equation of the line is given as

$$ax + by = c \quad (c > 0), \tag{1}$$

we have first to throw this into the "perpendicular" form.

Suppose it becomes

$$x \cos \alpha + y \sin \alpha = p. \tag{2}$$

Then, by equating the values we find from these two equations for the intercepts upon the axes, we obtain

$$\frac{\cos \alpha}{a} = \frac{\sin \alpha}{b} = \frac{p}{c}.$$

Therefore
$$c \cos \alpha = ap,$$
$$c \sin \alpha = bp$$

and
$$c^2 = (a^2 + b^2)p^2.$$
$$\therefore \; c = \sqrt{a^2 + b^2}\,p,$$

where there is no ambiguity in the square root, as both p and c are positive.

Hence
$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}},$$
$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

and
$$p = \frac{c}{\sqrt{a^2 + b^2}}.$$

Therefore the "perpendicular" form of the line

$$ax + by = c \quad (c > 0)$$

is
$$\frac{ax}{\sqrt{a^2 + b^2}} + \frac{by}{\sqrt{a^2 + b^2}} = \frac{c}{\sqrt{a^2 + b^2}}.$$

Hence the length of the perpendicular from (x_0, y_0) *upon*

$$ax + by - c = 0$$

is
$$\pm\left(\frac{ax_0 + by_0 - c}{\sqrt{a^2 + b^2}}\right).$$

And the positive sign is taken when (x_0, y_0) is upon the opposite side of the line from the origin, the negative sign when it is on the same side of the line as the origin.*

This result holds for the equation of the straight line, in whatever form it is given. The reason for the change of sign in the expression for the length of the perpendicular is that the line $lx+my+n=0$ divides the plane of xy into two parts. In one of these the expression $lx+my+n$ is positive; and in the other it is negative. Upon the line the expression vanishes.

Ex. 1. Transform the equations

(i.) $3x \pm 4y = 5$, (ii.) $3x \pm 4y = -5$

into the perpendicular form, and write down the value of α for each with the help of the Trigonometrical Tables.

2. Write down the length of the perpendicular from the origin upon the line joining (2, 3), (6, 7).

3. Write down the length of the perpendicular from the point (2, 3) upon the lines $4x+3y=7$, $5x+12y=20$, $3x+4y=8$.

4. Find the inscribed and escribed centres of the triangle whose sides are

$$3x+4y=0, \quad 5x-12y=0, \quad y=15,$$

and the equations of the internal and external bisectors of the angles of this triangle, distinguishing the different lines.

[A fuller discussion of the subject matter of this chapter is given in such books on Analytical Geometry as (i.) Briggs and Bryan's *Elements of Co-ordinate Geometry*, Part I., Chapters i.-x.; (ii.) Loney's *Co-ordinate Geometry*, Chapters i.-vi.; (iii.) C. Smith's *Elementary Treatise on Conic Sections*, Chapters i. and ii.; and (iv.) Gibson and Pinkerton's *Elements of Analytical Geometry*, Chapters i.-v.

In all these books a large number of examples will be found illustrating the points we have discussed.]

EXAMPLES ON CHAPTER I

1. Find the equation of the locus of the point P which moves so that

$$\text{(i.) } AP^2+PB^2=c^2$$
$$\text{(ii.) } AP^2-PB^2=c^2$$
$$\text{(iii.) } AP \cdot PB=c^2,$$

A and B being the points $(-a, 0)$, $(a, 0)$.

---

* *Rule.*—To find the length of the perpendicular from a given point (x_0, y_0) upon a given straight line $lx+my+n=0$,

insert the values (x_0, y_0) in place of (x, y) in the linear expression and divide by the square root of the sum of the squares of the coefficients of x and y in this expression. *The absolute value of*

$$\frac{lx_0+my_0+n}{\sqrt{l^2+m^2}}$$

is the length of the perpendicular.

2. Find the equation of the straight line through $(-1, 3)$, $(3, 2)$, and show that it passes through $(11, 0)$.

3. Show that the lines

$$\begin{aligned} 3x - 2y + 7 &= 0 \\ 4x + y + 3 &= 0 \\ 19x + 13y &= 0 \end{aligned}$$

all pass through one point, and find its co-ordinates.

4. Find the equations of the lines through the origin parallel and perpendicular to the lines of Ex. 3; also those through the point $(2, 2)$.

5. Find the equation of the line joining the feet of the perpendiculars from the origin upon the lines

$$\begin{aligned} 4x + y &= 17 \\ x + 2y &= 5. \end{aligned}$$

6. Draw the lines

$$\begin{aligned} 4y + 3x &= 12 \\ 3y + 4x &= 24. \end{aligned}$$

Find the equations of the bisectors of the angles between them, distinguishing the two lines.

7. The sides of a triangle are

$$\begin{aligned} x - y + 1 &= 0 \\ x - 4y + 7 &= 0 \\ x + 2y - 11 &= 0. \end{aligned}$$

Find (i.) the co-ordinates of its angular points,
(ii.) the tangents of its angles,
(iii.) the equations of the internal and external bisectors of these angles.

8. The angular points of a triangle are at $(0, 0)$, $(2, 4)$, $(-6, 8)$. Find
(i.) the equations of the sides,
(ii.) the tangents of the angles,
(iii.) the equations of the medians,
(iv.) the equations and lengths of the perpendiculars from the angular points on the opposite sides,
(v.) the equations of the lines through the angular points parallel to the opposite sides,
(vi.) the co-ordinates of the C.G.,
(vii.) the co-ordinates of the centres of the inscribed, circumscribed and nine-points circles.

9. Prove that the area of a triangle whose angular points are the origin, (x_1, y_1) and (x_2, y_2), is equal to the absolute value of $\frac{x_1y_2 - y_2x_1}{2}$; and find the corresponding expression for the area when the angular points are (x_1, y_1), (x_2, y_2) and (x_3, y_3).

10. Find the areas of the triangles given in Exs. 7 and 8.

CHAPTER II

THE MEANING OF DIFFERENTIATION

§ 11. The Idea of a Function.

If two variable quantities are related to one another in such a way that to each value of the one corresponds a definite value of the other, the one is said to be a function of the other. The variables being x and y, we express this by the equation $y=f(x)$. In this case x and y are called the independent and dependent variables respectively. Analytical Geometry furnishes us with a representation of such functions of great use in the experimental sciences. The variables are taken as the coordinates of a point, and the curve, whose equation is

$$y=f(x),$$

gives us a picture of the way in which the variables change.

In these chapters we shall assume that the equation $y=f(x)$ gives us a curve. There are, however, some peculiar functions which cannot thus be represented.

§ 12. Examples from Physics and Dynamics.

If a quantity of a perfect gas is contained in a cylinder closed by a piston, the volume of the gas will alter with the pressure upon the piston. Boyle's Law expresses the relationship between the pressure p upon unit area of the piston, and the volume v of the gas, when the temperature remains unaltered. This law is given by the equation

$$pv=p_0v_0,$$

where p_0, v_0 are two corresponding values of the pressure and the volume. When the volume v for unit pressure is unity, this equation becomes

$$pv = 1,$$

and the rectangular hyperbola, whose equation is

$$xy = 1,$$

will show more clearly than any table of numerical values of p and v the way in which these quantities change.

When the pressure is increased past a certain point Boyle's Law ceases to hold, and the relation between p and v in such a case is given by van der Waals's equation :—

$$\left(p + \frac{a}{v^2}\right)(v - b) = c,$$

a, b and c being certain positive quantities which have been approximately determined by experiment for different gases. Inserting the values of a, b and c for the gas under consideration, and drawing the curve

$$\left(x + \frac{a}{y^2}\right)(y - b) = c,$$

with suitable scales for x and y, the way in which p and v vary is made evident.

Such illustrations could be indefinitely multiplied. We add only two, taken from the case of the motion of a particle in a straight line.

When the *velocity is constant*, the distance s from a fixed point in the line to the position of the particle at time t is given by

$$s = vt + s_0,$$

where s_0 is the distance to the initial position of the particle, and v is the constant velocity.

The straight line $\qquad s = vt + s_0$

represents the relation between s and t, the co-ordinates now being referred to axes of s and t.

When the *acceleration is constant*, the corresponding equation is

$$s = \tfrac{1}{2}ft^2 + v_0t + s_0,$$

where $\qquad f =$ the acceleration,

$\qquad v_0 =$ the initial velocity

and $\qquad s_0 =$ the distance to the initial position.

In this case we have the parabola

$$s = \tfrac{1}{2} ft^2 + v_0 t + s_0$$

in the s, t diagram.

Also in both these cases we might obtain an approximate value of s for a given value of t, or an approximate value of t for a given value of s, by simple measurements in the figures representing the respective curves.

§ 13. The Fundamental Problem of the Differential Calculus.

The aim of the Differential Calculus is the investigation of the rate at which one variable quantity changes with regard to another, when the change in the one depends upon the change in the other, and the magnitudes vary in a continuous manner. Of course there are also cases in which the variable we are examining depends upon more than one variable. However, to such cases only a passing reference can be made in this book.

The element of time does not necessarily enter into the idea of a rate, and we may be concerned with the rate at which the pressure of a gas changes with the volume, or the length of a metal rod with the temperature, or the temperature of a conducting wire with the strength of the electric current along it, or the boiling point of a liquid with the barometric pressure, or the velocity of a wave with the density of the medium, or the cost of production of an article with the number produced, etc. etc. The simplest cases of rates of change are, however, those in which time does enter, and we shall begin our consideration of the subject with such examples.

§ 14. Rectilinear Motion.

In elementary dynamics the velocity of a point, which is moving uniformly, is defined as its rate of change of position, and this is equal to the quotient obtained by dividing the distance traversed in any period by the duration of the period, the distance being expressed in terms of a unit of length, and the period in terms of some unit of time.

When equal distances are covered in equal times this fraction is a perfectly definite one and does not depend upon the time, but when the rate of change of position is gradually altering, as, for instance, in the case of a body falling under gravity, the value of such a fraction alters with the length of the time considered. If, however, we note the distance travelled in different intervals measured from the time t, such intervals being taken smaller and smaller, we find that the values we obtain for what we might call the *average velocity* in these intervals are getting nearer and nearer to a definite quantity.

For example, in the case of a body falling from rest the distance fallen and the time are connected by the following equation,

$$s = \tfrac{1}{2}gt^2.$$

Let us fix upon a certain time t and the distance s which corresponds to that time.

Let $(s + \delta s)$ be the distance which corresponds to the time $(t + \delta t)$.

These quantities δs and δt added to s and t are called the "increments" of these variables.*

Then $$s + \delta s = \tfrac{1}{2}g(t + \delta t)^2 = \tfrac{1}{2}gt^2 + gt(\delta t) + \tfrac{1}{2}g(\delta t)^2.$$

Therefore $$\frac{\delta s}{\delta t} = gt + \tfrac{1}{2}g(\delta t).$$

Now let t be kept fixed, but let the increment δt get smaller and smaller.

It is clear that as δt tends to zero, the *average velocity*

$$gt + \tfrac{1}{2}g(\delta t),$$

for the interval t to $(t + \delta t)$, approaches nearer and nearer to the value gt.

This value towards which the average velocity tends as the interval diminishes is called the *velocity at the instant* t, on the understanding that we can get an "average velocity" as near this as we please by taking the interval sufficiently small.

The actual motion with these average velocities in the successive intervals would be a closer and closer approximation to the continually changing motion in proportion to the minuteness of the subdivisions of the time. The advantage of the method of the Differential Calculus is that it gives us a means of getting these "instantaneous velocities," or rates of change, at the time considered. When the mathematical formula connecting the

---

* *When these "increments" are small, it is convenient to speak of them as "the little piece added to s" and "the little piece added to t." It has to be noticed that the symbols δs and δt have to be taken as a whole. The beginner is apt to look upon δs as δ × s, when he uses it in an algebraical expression.*

quantities is given, we can state what the rate of change of the one is with regard to the other, without being dependent upon an approximation obtained by a set of observations in gradually diminishing intervals.

§ 15. **Limits. Differential Coefficient.**

If a variable which changes according to some law can be made to approach some fixed constant value as nearly as we please, but can never become exactly equal to it, the constant is called the limit of the variable under these circumstances. Now if this variable is x, and the limiting value of x is a, the dependent variable y (where $y=f(x)$) may become more and more nearly equal to some fixed constant value b as x tends to its limit a, and we may be able to make y differ from b by as little as we please, by making x get nearer and nearer to a. In this case b is called the limit of the function as x approaches its limit a, or more shortly, the limit of the function for $x=a$.

As the variable x is only supposed gradually to tend towards the value a, without actually attaining that value, it is better to write this in the form
$$\underset{x \to a}{\text{Lt}}\ (y)=b,$$
rather than in the form
$$\underset{x=a}{\text{Lt}}\ (y)=b.$$

In this way we emphasize the fact that it is not the value of y for x *equal* to a with which we are dealing. What we are concerned with is the limiting value of y as x *converges* to a as its limit.

Ex. (i.) If
$$y=\frac{\sin x}{x},$$
$$\underset{x \to 0}{\text{Lt}}\ (y)=1.$$

(ii.) If
$$y=x \log_{10} x,$$
$$\underset{x \to 0}{\text{Lt}}\ (y)=0.$$

(iii.) If
$$y=\frac{1}{x},$$
$$\underset{x \to 0}{\text{Lt}}\ (y)=\infty,$$
or, more correctly, y has no limit as x tends to zero.*

* For a fuller elementary discussion of the idea of a limit, see Love's *Elements of the Differential and Integral Calculus*, Ch. II., §§ 19, 20 and Appendix.

The subject is also discussed in such standard text-books as Lamb's, Gibson's, and Osgood's.

In this last example the function increases without limit as x approaches its limit. We might have the corresponding case of x increasing without limit and the function having a definite limit: *e.g.* if

$$y = a^x \text{ where } 0 < a < 1,$$

$$\operatorname*{Lt}_{x \to \infty} (y) = 0.$$

This idea of a limit has already (§ 14) been employed, and when $s = \frac{1}{2}gt^2$, the velocity at the time t of the moving point is what we now denote by the symbol

$$\operatorname*{Lt}_{\delta t \to 0} \left(\frac{\delta s}{\delta t}\right).$$

In the general case, when the relation between s and t is $s = f(t)$, we take the distance at the time $(t + \delta t)$ as $(s + \delta s)$.

Then we have $$s + \delta s = f(t + \delta t)$$

and $$\frac{\delta s}{\delta t} = \frac{f(t + \delta t) - f(t)}{\delta t}.$$

Hence the velocity at the time t is given by

$$v = \operatorname*{Lt}_{\delta t \to 0} \left(\frac{\delta s}{\delta t}\right) = \operatorname*{Lt}_{\delta t \to 0} \left\{\frac{f(t + \delta t) - f(t)}{\delta t}\right\}.$$

The limiting value of the ratio of the increment of s to the increment of t, as the increment of t approaches zero, is called the differential coefficient of s with regard to t. Instead of writing $\operatorname*{Lt}_{\delta t \to 0} \left(\frac{\delta s}{\delta t}\right)$, *we use the symbol* $\frac{ds}{dt}$ *for this limiting value.*

It must, however, be carefully noticed that in this symbol ds and dt cannot, so far as we are here concerned, be taken separately, and that $\frac{ds}{dt}$ *stands for the result of a definite mathematical operation, namely, the evaluation of the limiting value of the ratio of the corresponding increments of s and t, as the increment of t converges to zero.**

We shall see later, in § 38, that there is another notation in which ds and dt are spoken of as separate quantities, but until that section is reached, it will be well always to think of the differential coefficient as the result of the operation we have just described.

* For this and other reasons we shall often write $\frac{d}{dt}f(t)$, instead of $\frac{d\,f(t)}{dt}$. This is also written $f'(t)$.

It is clear that if δt is very small, the corresponding increment of s, namely δs, will be very approximately given by $\frac{ds}{dt}\delta t$. Still it is not a true statement, but only an approximation, to say that in this case

$$\delta s = \frac{ds}{dt}\delta t.$$

However, this approximation is very important. It may be employed in finding the change in the dependent variable due to a *small* change in the independent variable, or the error in the evaluation of a function due to a *small* error in the determination of the variable, provided we know the differential coefficient of the function.

We add some examples in which the differential coefficients are to be obtained from the above definition, viz.—

If $$s = f(t),\quad \frac{ds}{dt} = \underset{\delta t \to 0}{\text{Lt}} \left\{ \frac{f(t+\delta t) - f(t)}{\delta t} \right\}.$$

Ex. 1. If $s = at + b$, $\frac{ds}{dt} = a$.

2. If $s = at^2 + 2bt + c$, $\frac{ds}{dt} = 2(at + b)$.

3. If $\theta = \omega t$, $\frac{d\theta}{dt} = \omega$.

4. If $y = mx + n$, $\frac{dy}{dx} = m$.

5. If $y = ax^2$, $\frac{dy}{dx} = 2ax$.

§ 16. **Geometrical Illustration of the Meaning of the Differential Coefficient.**

The gradient, or slope, of a straight line has been defined in § 5. The gradient of a curve at any point is the gradient of the tangent at that point.

We obtain another illustration of the meaning of the differential coefficient by considering the gradient, or slope, of the curve

$$y = f(x).$$

Let P be a certain point (x, y), which we suppose fixed.

Let Q be another point, its co-ordinates being denoted by $(x + \delta x, y + \delta y)$.

Let the tangent at P make an angle ϕ with Ox.

Then, in Fig. 7,

$$\left.\begin{aligned} OM &= x \\ ON &= x + \delta x \\ MN &= \delta x \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} MP &= y = f(x) \\ MQ &= (y + \delta y) = f(x + \delta x) \\ HQ &= \delta y \quad = f(x + \delta x) - f(x). \end{aligned}\right\}$$

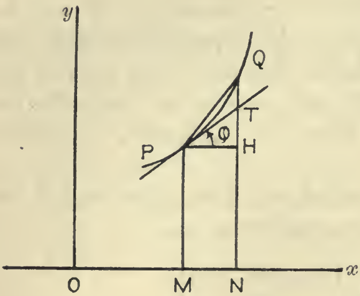


FIG. 7.

Thus the slope of the secant PQ

$$= \tan HPQ$$
$$= \frac{\delta y}{\delta x}$$
$$= \frac{f(x + \delta x) - f(x)}{\delta x}.$$

Now, if we keep P fixed, and let Q approach P, the secant PQ gets nearer and nearer the tangent at P, and the limiting value of the fraction $\frac{\delta y}{\delta x}$, as δx gets smaller and smaller, is $\tan \phi$.*

Thus, with the same notation as before,

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\delta y}{\delta x}\right) = \underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{f(x + \delta x) - f(x)}{\delta x}\right) = \tan \phi.$$

We have therefore shown that when the dependent and independent variable are the ordinate and abscissa of a point upon a curve, the differential coefficient is equal to the gradient of the curve.

Since the slope of the tangent is known when $\frac{dy}{dx}$ *is found, we can write down the equation of the tangent at a point* (x_0, y_0) *on the curve* $y = f(x)$, *when the value of* $\frac{dy}{dx}$ *at that point is known.*

This equation is [cf. § 6, (B)]

$$\frac{y - y_0}{x - x_0} = \text{the value of } \frac{dy}{dx} \text{ at } x = x_0.$$

* The increments δx, δy need not be positive. Unless the curve has a sharp corner at the point considered, the limiting position of the secant PQ would be the same whether δx were positive or negative.

Writing $f'(x)$ for the differential coefficient of $f(x)$, and $f'(x_0)$ for the value of $f'(x)$ when x has the value x_0, this equation becomes

$$y - y_0 = (x - x_0)f'(x_0).$$

Ex. Find the value of $\frac{dy}{dx}$ at the point (2, 1) on the parabola $4y = x^2$, and show that the equation of the tangent at that point is

$$x - y = 1.$$ [Cp. p. 83.]

§ 17. **Approximate Graphical Determination of the Differential Coefficient.**

When the equation connecting x and y is such that the curve

$$y = f(x)$$

may be easily drawn, the slopes of the various positions of the secant PQ, as Q is made to move nearer and nearer to P, will give a series of values more and more nearly approximating to the value of $\frac{dy}{dx}$ at that point. An instructive example is the case of the curve

$$y = x^2,$$

in which the following table of values of δx, δy and $\frac{\delta y}{\delta x}$ can readily be obtained. The way in which $\frac{\delta y}{\delta x}$ approaches its limiting value 2 at the point where $x = 1$ is evident.

δx	1	·9	·8	·7	·6	·5	·4	·3	·2	·1	·09	·08	·07	·06	·05	·04	·03	·02	·01
δy	3	2·61	2·24	1·89	1·56	1·25	·96	·69	·44	·21	·1881	·1664	·1449	·1236	·1025	·0816	·0609	·0404	·0201
$\frac{\delta y}{\delta x}$	3	2·9	2·8	2·7	2·6	2·5	2·4	2·3	2·2	2·1	2·09	2·08	2·07	2·06	2·05	2·04	2·03	2·02	2·01

§ 18. **Repeated Differentiation.**

We have now seen what is meant by the differential coefficient of a function of a single variable. The process of obtaining the differential coefficient is called differentiating the function. In the chapters which immediately follow we shall show how to differentiate the most important functions, and we shall prove some general theorems in differentiation. These will allow us to extend very widely the class of function for which we can write down the differential coefficients.

It is immaterial what symbols we use for the dependent and independent variables. We began by using s and t in the dynamical illustration of a rate of change. Then we used the relation $y=f(x)$, and found that the differential coefficient of y with regard to x was the slope of the curve $y=f(x)$ at the point (x, y). We shall use this geometrical notation most frequently, since one of the best introductions to the Calculus is through its applications in Analytical Geometry.

The differential coefficient $\frac{dy}{dx}$, or $f'(x)$, is itself a function of x, and we may differentiate this function. Its differential coefficient is written $\frac{d^2y}{dx^2}$, or $f''(x)$, and is called "the second differential coefficient" of y, or of $f(x)$.

This process may be repeated indefinitely. The differential coefficient of the second differential coefficient being called the third differential coefficient, and being written $\frac{d^3y}{dx^3}$, or $f'''(x)$, etc.

From this point of view $\frac{dy}{dx}$, or $f'(x)$, is called "the first differential coefficient."

Consider the case $\quad y=mx+n.$

We know from Chapter I. that this is the equation of a straight line of gradient m.

Therefore we have $\quad \frac{dy}{dx}=m.$

Also as the gradient m is the same for all values of x, its rate of change is zero.

Therefore $\quad \frac{d^2y}{dx^2}=0.$

Again take the case $\quad y=x^2.$

We have already seen how to differentiate such a function (cf. §§ 14, 15) proceeding from the definition of the differential coefficient. Later we shall obtain a rule, which will enable us to write down the answer immediately.

With the method already employed, we begin with the value x, and we have
$$y=x^2.$$

Then we take δx for the increment of x, and we write δy for the corresponding increment of y.

Therefore we have $\quad y+\delta y=(x+\delta x)^2.$

From these two equations it follows that

$$\delta y = 2x(\delta x) + (\delta x)^2.$$

$$\therefore \quad \frac{\delta y}{\delta x} = 2x + (\delta x).$$

$$\therefore \quad \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta y}{\delta x}\right) = 2x.$$

Thus $$\frac{dy}{dx} = 2x.$$

To find the second differential coefficient, we have to differentiate the expression for $\frac{dy}{dx}$.

In this case we find at once by calculation, or from our knowledge of the graph of $2x$, that

$$\frac{d^2y}{dx^2} = 2.$$

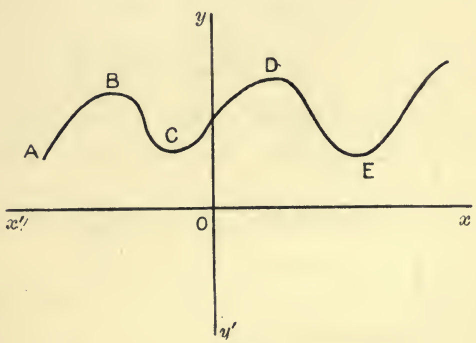


FIG. 8.

From Fig. 8, *it is obvious that, when* $\frac{dy}{dx}$ *is positive, the tangent is inclined at an acute angle to the axis of* x*: that, when it is negative, this angle is obtuse. A positive* $\frac{dy}{dx}$ *means that* y *increases with* x *at that point: a negative* $\frac{dy}{dx}$ *means that* y *diminishes as* x *increases. When* $\frac{dy}{dx}$ *vanishes, the tangent must be parallel to the axis of* x.

Let us imagine the curve ABC... to stand for a road, and that a traveller is marching along it in the positive direction of the axis of x, which is horizontal. When the traveller ascends, $\frac{dy}{dx}$ is positive: when he descends, $\frac{dy}{dx}$ is negative; and if the road is rounded off and no sudden changes of gradient occur, when he ceases to ascend and begins to descend, or the reverse, $\frac{dy}{dx}$ changes sign by passing through zero. [See also p. 71.]

What information can we obtain from the second differential coefficient of y regarding the curve $y=f(x)$?

We have seen that $\frac{d^2y}{dx^2}$ stands for the rate of change of the gradient. It follows that along the parts of the curve where the gradient is increasing, $\frac{d^2y}{dx^2}$ is *positive.* Also that along the parts of the curve where the gradient is diminishing, $\frac{d^2y}{dx^2}$ is *negative.*

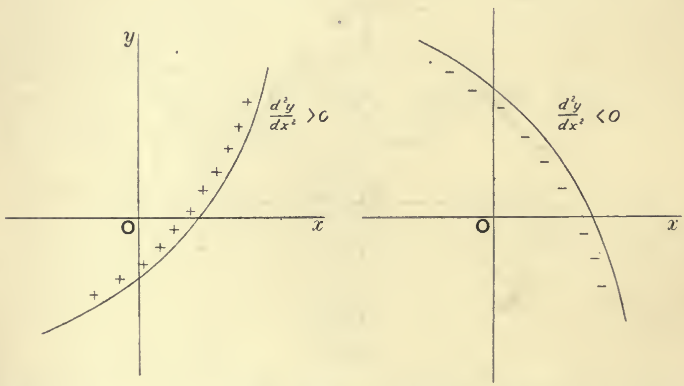


FIG. 9.

This can also be put in the following way: $\frac{d^2y}{dx^2}$ is *positive*, when the curve $y=f(x)$ is *concave upwards*; that is, concave, when looked at from above. Also $\frac{d^2y}{dx^2}$ is *negative*, when the curve is *convex upwards*; that is, convex, when looked at from above.

The second differential coefficient has also an important application in Dynamics.

The acceleration of a moving point is defined as its rate of change of velocity.

Let us return to the case of the motion of a point along a straight line, in which the distance and the time are connected by the relation

$$s=f(t).$$

Let the velocity at the time t be v.

Then we know that $$v=\frac{ds}{dt}=f'(t).$$

Also the acceleration at the time t is the rate of change of the velocity at that time.

Thus $$\text{the acceleration}=\frac{dv}{dt}$$
$$=\frac{d^2s}{dt^2}, \text{ or } f''(t).$$

E.g. If $$s=\tfrac{1}{2}gt^2,$$
$$v=\frac{ds}{dt}=gt;$$

and $$\text{the acceleration}=\frac{d^2s}{dt^2}=g.$$

Again, when the velocity is *increasing*, the acceleration is *positive*; when the velocity is *decreasing*, the acceleration is *negative.*

Therefore the sign of the second differential coefficient, $\frac{d^2s}{dt^2}$, tells us whether the velocity is increasing or decreasing at the instant considered. We shall return to this question later, and we shall see that when the second differential coefficient $\frac{d^2s}{dt^2}$ vanishes for a certain value of t, and is positive just before that value of t, and negative just after it, then at that particular instant the velocity has a maximum value. Also that when the change of sign is from negative to positive, the velocity has a minimum value at that time. [Cf. § 38.]

EXAMPLES ON CHAPTER II

The differential coefficients required in the examples on this chapter are to be obtained from the definition.

1. Plot the curves (i.) $y=x+x^2$ (ii.) $y=x^3$, and show that they have the same gradient when $x=1$.

2. By considering the area of a square and the volume of a cube, show that the differential coefficients of x^2 and x^3 are $2x$ and $3x^2$ respectively.

3. Show that the curves $y=x^2$ and $y=x^4$ intersect at the origin and the points (1, 1), $(-1, 1)$, and that at each of the two latter points the angle between the tangents is $\tan^{-1}\frac{2}{9}$.

4. Show that the gradient of the curve $y=x^3-3x$ at the point where $x=2$ is 9. Find the equation of the tangent there and trace the curve.

5. Find where the ordinate of the curve $y=3x-4x^2$ increases at the same rate as the abscissa, and where it decreases five times as fast as the abscissa increases.

6. If $s=ut-\frac{1}{2}gt^2$, find the values of the velocity and acceleration at the time t.

7. A cylinder has a height h ins. and a radius r ins.; there is a possible small error δr in r. Find an approximate value of the possible error in the computed volume.

8. Find approximately the error made in the volume of a sphere by making a small error δr in the radius r. The radius is said to be 20 ins.; give approximate values of the errors made in the computed surface and volume if there be an error of ·1 in. in the length assigned to the radius.

9. The area of a circular plate is expanding by heat. When the radius passes through the value 2 ins. it is increasing at the rate of ·01 in. per sec. Show that the area is increasing at the rate of $\cdot 04\pi$ sq. in. per sec. at that time.

10. The length of a bar at temperature 0° is unity. At temperature $t°$ its length l is given by the equation

$$l=1+at+bt^2;$$

find the rate at which the bar increases in length at temperature $t°$, and give an approximation to the increase in length due to a small rise in temperature.

11. If the diameter of a spherical soap-bubble increases uniformly at the rate of ·1 centimetre per second, show that the volume is increasing at the rate of $\cdot 2\pi$ cub. cent. per second when the diameter becomes 2 centimetres.

12. A ladder 24 feet long is leaning against a vertical wall. The foot of the ladder is moved away from the wall, along the horizontal surface of the ground and in a direction at right angles to the wall, at a uniform rate of 1 foot per second. Find the rate at which the top of the ladder is descending on the wall, when the foot is 12 feet from the wall.

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS; AND SOME GENERAL THEOREMS ON DIFFERENTIATION

§ 19. **The Differentiation of $\mathbf{x^n}$, when n is a Positive Integer.**

We have already seen that,

$$\text{when} \quad y = x^2,$$

$$\frac{dy}{dx} = 2x.$$

A similar argument would show us that,

$$\text{when} \quad y = x^3,$$

$$\frac{dy}{dx} = 3x^2;$$

and that,

$$\text{when} \quad y = x^4,$$

$$\frac{dy}{dx} = 4x^3.$$

These suggest that

$$\textbf{when} \quad \mathbf{y = x^n},$$

$$\mathbf{\frac{dy}{dx} = nx^{n-1}}.$$

As a matter of fact this formula is true, when n is any number independent of x. However we shall prove it, at present, only for the case of *n a positive integer.* The cases when the index of the power of x is a fraction or negative we shall examine later.*

* In the first edition of this book the usual proof of this theorem is given, the Binomial Theorem for any index being employed. The student, who understands the use of Infinite Series, will probably prefer that proof, but it seems better to give those who have not read that difficult portion of Algebra, or have not properly understood it, an alternative method. Similar changes are made in the discussion of the differentiation of the exponential and logarithmic functions, and our subject is developed without the use of the Theory of Infinite Series at all. The proofs referred to are given in the Appendix (p. 129).

As usual, we begin with the value x, and we put

$$y = x^n. \qquad (1)$$

Then we take the increment δx of x, and we write δy for the corresponding increment of y.

It follows that $\qquad y + \delta y = (x + \delta x)^n.$

Now we know from Elementary Algebra that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \dots + b^n,$$

when n is a positive integer.

Therefore we have

$$y + \delta y = x^n + nx^{n-1}(\delta x) + \frac{n(n-1)}{1.2}x^{n-2}(\delta x)^2 + \dots + (\delta x)^n. \qquad (2)$$

From (1) and (2) we have

$$\delta y = nx^{n-1}(\delta x) + \frac{n(n-1)}{1.2}x^{n-2}(\delta x)^2 + \dots + (\delta x)^n.$$

Therefore

$$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{1.2}x^{n-2}(\delta x) + \dots + (\delta x)^{n-1}.$$

Now n is a definite positive integer, and there are $(n-1)$ terms in this expression after the first. All of these terms have the factor δx. If we let δx tend to zero, the sum of these terms must vanish in the limit.

It follows that $\qquad \underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\delta y}{\delta x}\right) = nx^{n-1}.$

Thus we have proved that, when n is a positive integer, the differential coefficient of x^n is nx^{n-1}.

Ex. Fill up the blank column in the following table:

$f(x)$.	$f'(x)$.
x	
x^2	
x^3	
x^4	
x^5	
x^6	
x^7	
x^8	
x^9	
x^{10}	

§ 20. **General Theorems on Differentiation.**

Before proceeding to obtain the differential coefficients of other functions, it will be useful to show that many complicated expressions can be differentiated by means of this result, with the help of the following general theorems:—

PROPOSITION I. *Differentiation of a Constant.*

It is clear that, if $y=a$, the slope of the line is zero, and $\frac{dy}{dx}=0$. In other words, it is obvious that if a magnitude remains the same its rate of change is zero.

Thus *the differential coefficient of a constant is zero.*

PROPOSITION II. *Differentiation of the Product of a Constant and a Function of x.*

Let $y=au$, where a is a constant, and u is a function of x.

We begin with the value x, and we take δx for the increment of x.

When x becomes $x+\delta x$, let u become $u+\delta u$, and y become $y+\delta y$.

Then
$$y+\delta y=a(u+\delta u),$$
and
$$\frac{\delta y}{\delta x}=a\frac{\delta u}{\delta x}.$$

For the value of x considered, we are supposed to know that $\frac{du}{dx}$ exists; in other words, that

$$\underset{\delta x\to 0}{\mathrm{Lt}}\left(\frac{\delta u}{\delta x}\right)$$

is a definite number.

It follows that
$$\underset{\delta x\to 0}{\mathrm{Lt}}\left(\frac{\delta y}{\delta x}\right)=a\underset{\delta x\to 0}{\mathrm{Lt}}\left(\frac{\delta u}{\delta x}\right).$$

Therefore
$$\frac{dy}{dx}=a\frac{du}{dx}.$$

Thus the differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.

The geometrical meaning of this theorem is that if all the ordinates of a curve are increased in the same ratio, the slope of the curve is increased in the same ratio.

The dynamical meaning will be obvious to the reader.

PROPOSITION III. *Differentiation of a Sum.*

Let $$y = u + v.$$

Then, as before, $$y + \delta y = (u + \delta u) + (v + \delta v),$$

and $$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}.$$

We are supposed to know that, for the value of x considered, $\frac{\delta u}{\delta x}$ and $\frac{\delta v}{\delta x}$ have definite limiting values as $\delta x \to 0$.

It follows, on proceeding to the limit, that

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

The same argument applies to the sum (or difference) of several functions, and we see that the *differential coefficient of such a sum is the sum of the several differential coefficients.*

Ex. Differentiate the following functions :—

(i.) $x(2+x)^2$

(ii.) $(a+bx+cx^2)x$

(iii.) $\frac{x^4}{4}+\frac{x^3}{3}+\frac{x^2}{2}+x+1$

(iv.) $2+2x+3x^2$.

PROPOSITION IV. *Differentiation of the Product of Two Functions.*

Let $$y = uv.$$

Then, as before, $$y + \delta y = (u + \delta u)(v + \delta v).$$

Thus $$\delta y = v\,\delta u + u\,\delta v + \delta u\,\delta v,$$

and $$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \delta u\frac{\delta v}{\delta x}.$$

We are supposed to know that, for the value of x considered, $\frac{\delta u}{\delta x}$ and $\frac{\delta v}{\delta x}$ have definite limiting values as $\delta x \to 0$.

In this case $\delta u \to 0$, as $\delta x \to 0$. It follows, on proceeding to the limit, that

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

This result may be written

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx};$$

and when $y = uvw$, we would obtain in the same way,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}. \quad \text{(Cf. § 31.)}$$

In the case of two functions it is easy to remember that *the differential coefficient of the product of two functions is equal to the first function × the differential coefficient of the second + the second function × the differential coefficient of the first.*

Ex. Differentiate the following functions as products :—

(i.) $(1+x^2)(2x^2-1)$

(ii.) $(2x^2+1)(x+2)^2$

(iii.) $(ax+b)^2(cx+d)^2$

(iv.) $x(x+1)(x+2)$,

and show that the results are the same if the expressions are multiplied out and then differentiated.

PROPOSITION V. *Differentiation of a Quotient.*

Let $$y = u/v.$$

Then, as before, $$y + \delta y = \frac{u+\delta u}{v+\delta v},$$

and $$\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{v\,\delta u - u\,\delta v}{v^2\left(1+\dfrac{\delta v}{v}\right)}.$$

Therefore $$\frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v^2\left(1+\dfrac{\delta v}{v}\right)}.$$

We are supposed to know that, for the value of x considered, $\frac{\delta u}{\delta x}$ and $\frac{\delta v}{\delta x}$ have definite limiting values as $\delta x \to 0$.

In this case $\delta v \to 0$, as $\delta x \to 0$.

Proceeding to the limit, it follows that

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.*$$

---

* This result may be obtained by writing

$$vy = u,$$

and then differentiating both sides of the equation.

In words, *to find the differential coefficient of a quotient, from the product of the denominator and the differential coefficient of the numerator subtract the product of the numerator and the differential coefficient of the denominator, and divide the result by the square of the denominator.*

We can use this result to find the differential coefficient of x^n, when n is a *negative* integer.

Let $n = -m$, where m is a positive integer.

Then we have $$y = \frac{1}{x^m}.$$

Therefore $$\frac{dy}{dx} = \frac{-mx^{m-1}}{x^{2m}},$$

since the differential coefficient of the numerator is zero, and the differential coefficient of the denominator is mx^{m-1}.

Thus $$\frac{dy}{dx} = -mx^{-m-1}.$$

In particular, if $$y = \frac{1}{x},$$

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

We return to this on p. 40.

Ex. Differentiate the following expressions :—

(i.) $\dfrac{x+1}{2-x}$ (ii.) $\dfrac{(x+1)(x+2)}{(x+3)}$ (iii.) $\dfrac{x}{(x+1)(x+2)}$

(iv.) $\dfrac{(x+1)^2}{(x+2)^3}$ (v.) $\dfrac{1+x^2}{1-x^2}$ (vi.) $\dfrac{ax^2+2bx+c}{ax^2-2bx+c}$.

These five formulæ, with the help of the result of § 19, enable us to differentiate a large number of expressions, but they do not apply directly to such cases as $(ax+b)^{10}$, $(ax^2+2bx+c)^{20}$, etc.

Each of the above expressions is a function of a function of x, and we proceed to prove another general theorem :—

PROPOSITION VI. *Differentiation of a Function of a Function.*

Let $$y = F(u),$$
where $$u = f(x)$$

$$\left(\begin{array}{ll} e.g. & y = u^{100}, \\ \text{where} & u = a^2 + x^2 \end{array}\right).$$

We begin with the value x, and we take δx for the increment of x.

Then when x is changed to $x+\delta x$,

let u become $u+\delta u$,

and y become $y+\delta y$;

the functions being such that for a small change in x we have a definite and small change both in u and y.

But $$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}\times\frac{\delta u}{\delta x}.$$

Now we are supposed to know that, for the value of x considered, $\frac{\delta u}{\delta x}$ has a definite limiting value as $\delta x \to 0$. In this case, when $\delta x \to 0$, $\delta u \to 0$. Also we are supposed to know that $\frac{\delta y}{\delta u}$ has a definite limiting value as $\delta u \to 0$.

It follows, on proceeding to the limit, that

$$\frac{\mathbf{dy}}{\mathbf{dx}}=\frac{\mathbf{dy}}{\mathbf{du}}\times\frac{\mathbf{du}}{\mathbf{dx}}.$$

It is important to notice that this rule

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

cannot be inferred from striking out the du's, as if the expressions were fractions. We have already laid stress on the fact that the differential coefficient $\frac{dy}{dx}$ is not to be looked upon as a fraction *dy divided by dx*.

COROLLARY. If we put $y=x$ in the above result, we obtain

$$\frac{dx}{du}\times\frac{du}{dx}=1.$$

It follows that $$\frac{dx}{du}=\frac{1}{\frac{du}{dx}}.$$

Altering this notation, we can say that

$$\frac{\mathbf{dy}}{\mathbf{dx}}\times\frac{\mathbf{dx}}{\mathbf{dy}}=1, \text{ and that } \frac{\mathbf{dy}}{\mathbf{dx}}=\frac{1}{\frac{\mathbf{dx}}{\mathbf{dy}}}.$$

This theorem could have been proved directly, using an argument very similar to the above.

Also it is instructive to see that it follows immediately from the geometrical interpretation of the differential coefficient.

Ex. 1. Differentiate $(x+1)^4$.

Let $y=(x+1)^4$.

Then we have $y=u^4$, where $u=x+1$.

But $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$.

It follows that
$$\frac{dy}{dx}=4u^3\times 1$$
$$=4(x+1)^3.$$

2. Differentiate $(2x+1)^6$.

Let $y=(2x+1)^6$.

Then we have $y=u^6$, where $u=2x+1$.

But $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$.

It follows that
$$\frac{dy}{dx}=6u^5\times 2$$
$$=12(2x+1)^5.$$

3. Differentiate $(1-3x)^{10}$.

Let $y=(1-3x)^{10}$.

Then we have $y=u^{10}$, where $u=1-3x$.

It follows that $\dfrac{dy}{dx}=10u^9\times -3=-30(1-3x)^9$.

4. Prove that if $y=(ax+b)^n$, and n is a positive integer,
$$\frac{dy}{dx}=na(ax+b)^{n-1}.$$

5. If $y=(x+1)^4(2x+1)^6$, find $\dfrac{dy}{dx}$.

We have, by the rule for differentiating a product,
$$\frac{dy}{dx}=(x+1)^4\frac{d}{dx}(2x+1)^6+(2x+1)^6\frac{d}{dx}(x+1)^4.$$

Using the results of Exs. 1 and 2 above,
$$\frac{dy}{dx}=(x+1)^4.12(2x+1)^5+(2x+1)^6.4(x+1)^3$$
$$=4(x+1)^3(2x+1)^5\{3(x+1)+(2x+1)\}$$
$$=4(x+1)^3(2x+1)^5(5x+4).$$

6. If $y=\dfrac{(x+1)^4}{(2x+1)^6}$, $\dfrac{dy}{dx}=-4\dfrac{(x+2)(x+1)^3}{(2x+1)^7}$.

7. If $y=(ax^2+2bx+c)^{10}$, $\dfrac{dy}{dx}=20(ax+b)(ax^2+2bx+c)^9$.

8. Fill up the blank form in the following table :

$f(x)$.	$f'(x)$.
$(1+x)^2$	
$(1-x)^2$	
$(1+x^2)^4$	
$(1-x^2)^4$	
$(a+bx)^6$	
$(a-bx)^6$	
$\left(1+\frac{x^2}{2}\right)^3$	
$(1+2x^2)^4$	

§ 21. **The Differentiation of x^n, when n is any positive or negative number.**

We have shown in § 19 that when n is a positive integer, the differential coefficient of x^n is nx^{n-1}.

Also, we have seen in § 20 that this result is true, when n is any *negative* integer.

We shall now prove that it is true in general: in other words, that **when n is any positive or negative number, the differential coefficient of x^n is nx^{n-1}.**

Case (i.). Let $n=\frac{p}{q}$, a positive fraction in its lowest terms, p and q being positive integers.

Then $x^{\frac{p}{q}}$ stands for the real positive q^{th} root of x^p.

Let $$y=x^{\frac{p}{q}}.$$

Then we have $$y^q=x^p.$$

The differential coefficients of both sides with regard to x must be equal.

Thus $$\frac{d}{dx}y^q=\frac{d}{dx}x^p=px^{p-1}.$$

But $$\frac{d}{dx}y^q=\frac{d}{dx}y^q\times\frac{dy}{dx},\quad \text{by § 20, Prop. VI.}$$

Therefore $$\frac{d}{dx}y^q=qy^{q-1}\times\frac{dy}{dx}.$$

Therefore we have

$$qy^{q-1}\frac{dy}{dx}=px^{p-1}.$$

Therefore $$\frac{dy}{dx}=\frac{p}{q}\frac{x^{p-1}}{y^{q-1}}$$

$$=\frac{p}{q}\frac{x^{p-1}}{\left(x^{\frac{p}{q}}\right)^{q-1}}$$

$$=\frac{p}{q}\frac{x^{p-1}}{x^{p-\frac{p}{q}}}$$

$$=\frac{p}{q}x^{\frac{p}{q}-1}.$$

Thus, when $$y=x^{\frac{p}{q}},$$

$$\frac{dy}{dx}=\frac{p}{q}x^{\frac{p}{q}-1}.$$

Therefore our formula holds when n is any positive fraction.

We have already proved that it holds for n a positive integer.

Therefore it is true for any positive number.

Case (ii.). There remains the case of a negative index, integral or fractional.

Let $$y=x^{-m}, \quad \text{where} \quad m>0.$$

Then we have $$y=u^{m}, \quad \text{where} \quad u=\frac{1}{x}.$$

But $$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

Also $$\frac{dy}{du}=mu^{m-1}, \text{ by Case (i.).}$$

And $$\frac{du}{dx}=-\frac{1}{x^2},$$

by the rule for differentiating a quotient. [Cf. p. 36.]

It follows that $$\frac{dy}{dx}=mx^{-m+1}\times-\frac{1}{x^2}$$

$$=-mx^{-m-1}.$$

Therefore the formula holds for any negative number.

Combining these results, our theorem may be stated as follows:—

If $$\mathbf{y=x^{n}, \quad \frac{dy}{dx}=nx^{n-1},}$$

when n is any positive or negative number.*

* It will be noticed that we have assumed that n is a rational number. The theorem is true also for irrational numbers.

Ex. 1. Fill up the blank column in the following table :—

$f(x)$.	$f'(x)$.
$\sqrt{x}$	
$\frac{1}{\sqrt{x}}$	
$x^{\frac{3}{2}}$	
$\frac{1}{x^{\frac{3}{2}}}$	
$x^{\frac{5}{2}}$	
$\frac{1}{x^{\frac{5}{2}}}$	
$\frac{1}{x^2}$	
$\frac{1}{x^3}$	
$\frac{1}{x^4}$	
$\frac{1}{x^5}$	
$\frac{1}{x^{99}}$	

2. If $y=\frac{1}{1-x}$, $\frac{dy}{dx}=\frac{1}{(1-x)^2}$.

We have $y=\frac{1}{u}$, where $u=1-x$. Thus $y=u^{-1}$, where $u=1-x$.

But
$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

It follows that
$$\frac{dy}{dx}=-u^{-2}\times -1$$
$$=\frac{1}{u^2}$$
$$=\frac{1}{(1-x)^2}.$$

3. If $y=\frac{1}{(3x+4)^3}$, find $\frac{dy}{dx}$.

We have $y=u^{-3}$, where $u=3x+4$.

It follows that
$$\frac{dy}{dx}=-\frac{3}{u^4}\times 3$$
$$=-\frac{9}{(3x+4)^4}.$$

4. If $y=(ax+b)^n$, and n is any positive or negative number, prove that
$$\frac{dy}{dx}=na(ax+b)^{n-1}.$$

5. If $y=\dfrac{(x+1)^2}{(3x+4)^3}$, find $\dfrac{dy}{dx}$.

We shall work this example by the product rule instead of the quotient rule. This method often saves dividing out by some factors.

By the product rule [cf. § 20, Prop. IV.],

$$\frac{dy}{dx}=(x+1)^2\frac{d}{dx}\frac{1}{(3x+4)^3}+\frac{1}{(3x+4)^3}\frac{d}{dx}(x+1)^2.$$

Now we have seen in Ex. 2 that

$$\frac{d}{dx}\frac{1}{(3x+4)^3}=-\frac{9}{(3x+4)^4}.$$

And it is easy to show that

$$\frac{d}{dx}(x+1)^2=2(x+1).$$

It follows that
$$\frac{dy}{dx}=-9\frac{(x+1)^2}{(3x+4)^4}+\frac{2(x+1)}{(3x+4)^3}$$
$$=-\frac{(x+1)(3x+1)}{(3x+4)^4}.$$

6. Fill up the blank column in the following table:—

$f(x)$.	$f'(x)$.
$\dfrac{1}{(2x+3)^4}$	
$\dfrac{1}{(2x-3)^4}$	
$\dfrac{1}{(1-x^2)^2}$	
$\dfrac{1}{(1+x^2)^2}$	
$1-\dfrac{1}{1-x}$	
$1+\dfrac{1}{1+x}$	
$1+\dfrac{1}{(1+x)}+\dfrac{1}{2(1+x)^2}$	
$1+\dfrac{1}{(1-x)}+\dfrac{1}{2(1-x)^2}$	
$\dfrac{1}{\sqrt{1-x}}$	
$\dfrac{1}{\sqrt{1+x}}$	
$\sqrt{a^2+x^2}$	
$\sqrt{a^2-x^2}$	

EXAMPLES ON CHAPTER III

1. Find $\frac{dy}{dx}$ in the following cases :—

(i.) $y=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^3$.

(ii.) $y=\sqrt{2ax-x^2}$.

(iii.) $y=\sqrt{(x+1)(x+2)}$.

(iv.) $y=(x+a)^p(x+b)^q$.

(v.) $y=\sqrt{\frac{1+x}{1-x}}$.

(vi.) $y=\frac{(a-x)^p}{(b-x)^q}$.

(vii.) $y=\frac{x^n}{a+bx^m}$.

(viii.) $y=(1+x^n)^m$.

(ix.) $y=\sqrt{x^2+a^2}+\sqrt{x^2-a^2}$.

(x.) $y=\frac{1}{\sqrt{x^2+a^2}}+\frac{1}{\sqrt{x^2-a^2}}$.

(xi.) $y=\frac{x^3}{\sqrt{(1-x^2)^3}}$.

(xii.) $y=\sqrt{\frac{1+x+x^2}{1-x+x^2}}$.

2. Find the gradient at the point (x_0, y_0) in the following curves :—

(i.) $y^2=4ax$.

(ii.) $x^2+y^2=a^2$.

(iii.) $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$.

(iv.) $2xy=c^2$.

3. Prove that the equations of the tangents at (x_0, y_0) to these curves are respectively

(i.) $yy_0=2a(x+x_0)$.

(ii.) $xx_0+yy_0=a^2$.

(iii.) $\frac{xx_0}{a^2}\pm\frac{yy_0}{b^2}=1$.

(iv.) $xy_0+yx_0=c^2$.

4. A boy is running on a horizontal plane in a straight line towards the base of a tower 50 yards high. How fast is he approaching the top, when he is 500 yards from the foot, and he is running at 8 miles per hour?

5. A light is 4 yards above and directly over a straight horizontal path on which a man six feet high is walking, at a speed of 4 miles per hour, away from the light.

Find (i.) The velocity of the end of his shadow;

(ii.) The rate at which his shadow is increasing in length.

6. A man standing on a wharf is drawing in the painter of a boat at the rate of 4 feet per second. If his hands are 6 feet above the bow of the boat, prove that the boat is moving at the rate of 5 feet per second, when it is 8 feet from the wharf.

7. A vessel is anchored in 10 fathoms of water, and the cable passes over a sheave in the bowsprit, which is 12 feet above the water. If the cable is hauled in at the rate of 1 foot per second, prove that the vessel is moving through the water at a rate of $1\frac{1}{4}$ feet per second, when there are 20 fathoms of cable out.

8. If a volume v of a gas, contained in a vessel under pressure p, is compressed or expanded without loss of heat, the law connecting the pressure and volume is given by the formula

$$pv^{\gamma}=\text{constant},$$

where γ is a constant.

Find the rate at which the pressure changes with the volume.

9. In Boyle's Law, where $pv=c^2$, show that $\frac{dv}{dp}=-\frac{c^2}{p^2}$. What does the negative sign in this expression mean?

10. In van der Waals's equation

$$\left(p+\frac{a}{v^2}\right)(v-b)=\text{constant}.$$

Prove that
$$\frac{dv}{dp}=-\frac{(v-b)}{\left(p-\frac{a}{v^2}+\frac{2ab}{v^3}\right)}.$$

CHAPTER IV

THE DIFFERENTIATION OF THE TRIGONOMETRIC FUNCTIONS

(The angles are supposed to be measured in Radians)

§ 22. **The Differential Coefficient of the Sine.**

We begin with the value x, and we put

$$y = \sin x.$$

Then we take δx as the increment of x, and write δy for the corresponding increment of y.

It follows that $\qquad y + \delta y = \sin (x + \delta x).$

Therefore
$$\delta y = \sin (x + \delta x) - \sin x$$
$$= 2 \cos \left(x + \frac{\delta x}{2}\right) \sin \frac{\delta x}{2}.$$

Therefore
$$\frac{\delta y}{\delta x} = \cos \left(x + \frac{\delta x}{2}\right) \left\{ \frac{\sin \left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}} \right\}.$$

Proceeding to the limit, and remembering that

$\underset{\theta \to 0}{\text{Lt}} \left(\frac{\sin \theta}{\theta}\right) = 1$, it follows that

If $\mathbf{y = \sin x}$, $$\frac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{\cos x}.$$

N.B.—When $y = \sin (mx + n)$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \text{ where } u = mx + n,$$
$$= \frac{d (\sin u)}{du}\frac{du}{dx}$$
$$= \cos u \times m$$
$$= m \cos (mx + n).$$

Ex. 1. Fill up the blanks in the following table :—

$f(x)$	$\sin 2x$	$2\sin\frac{x}{2}$	$\frac{1}{3}\sin 3x$	$\frac{3}{4}\sin(4x+5)$	$3\sin x - \sin 3x$	$\sin(1-x)$
$f'(x)$	2sin2x	cos x/2	cos 3x	3cos (4x+5)	3cos x -3cos3x	- cos (1-x)

2. Prove from the definition of $\frac{dy}{dx}$, that when $y=\sin(mx+n)$,

$$\frac{dy}{dx}=m\cos(mx+n).$$

§ 23. **The Differential Coefficient of the Cosine.**

We begin with the value x, and we put

$$y=\cos x.$$

Then we take δx as the increment of x, and we write δy for the corresponding increment of y.

It follows that $\quad y+\delta y=\cos(x+\delta x).$

Therefore
$$\delta y=\cos(x+\delta x)-\cos x$$
$$=-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}.$$

Thus
$$\frac{\delta y}{\delta x}=-\sin\left(x+\frac{\delta x}{2}\right)\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right).$$

Proceeding to the limit, it follows that

If $\mathbf{y=\cos x}$, $\quad \mathbf{\frac{dy}{dx}=-\sin x.}$

N.B.—When $y=\cos(mx+n)$, $\quad \frac{dy}{dx}=-m\sin(mx+n).$

Ex. 1. Fill up the blanks in the following table :—

$f(x)$	$\cos 2x$	$-2\cos\frac{x}{2}$	$\frac{1}{3}\cos 3x$	$1-\cos 2x$	$\cos 3x+3\cos x$	$\frac{1}{2}\cos(1-2x)$
$f'(x)$						

2. Prove from the definition of $\frac{dy}{dx}$, that when $y=\cos(mx+n)$,

$$\frac{dy}{dx}=-m\sin(mx+n).$$

§ 24. The Differential Coefficient of the Tangent.

Let $$y = \tan x = \frac{\sin x}{\cos x}.$$

Then $$\frac{dy}{dx} = \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x.$$

Thus, if $\mathbf{y = \tan x}$, $$\frac{\mathbf{dy}}{\mathbf{dx}} = \mathbf{\sec^2 x}.$$

N.B.—When $y = \tan(mx+n)$, $\frac{dy}{dx} = m\sec^2(mx+n)$.

Ex. 1. Fill up the blanks in the following table :—

$f(x)$	$2\tan\frac{x}{2}$	$\tan\sqrt{x}$	$\tan(x^2)$	$1+\frac{1}{3}\tan 3x$	$\tan 2(1-x)$
$f'(x)$					

2. Prove from the definition of $\frac{dy}{dx}$, that when $y = \tan(mx+n)$,

$$\frac{dy}{dx} = m\sec^2(mx+n).$$

From these three results it is easy to deduce the following :—

$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{\cot x = -\operatorname{cosec}^2 x}; \quad \frac{d}{dx}\cot(mx+n) = -m\operatorname{cosec}^2(mx+n).$$

$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{\sec x = \frac{\sin x}{\cos^2 x}}; \quad \frac{d}{dx}\sec(mx+n) = m\frac{\sin(mx+n)}{\cos^2(mx+n)}.$$

$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{\operatorname{cosec} x = -\frac{\cos x}{\sin^2 x}}; \quad \frac{d}{dx}\operatorname{cosec}(mx+n) = -m\frac{\cos(mx+n)}{\sin^2(mx+n)}.$$

§ 25. Geometrical Proofs of these Theorems.

All these cases of differentiation may be discussed geometrically. We take the case of the tangent. The reader is recommended to work out for himself the cases of the sine and cosine.

Let $\angle$MOP be the angle θ radians, and let OM be 1 unit in length.

Let $\angle$POQ be $\delta\theta$, and let QPM be perpendicular to the line OM from which θ is measured.

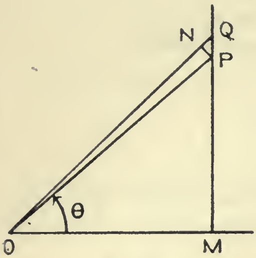


FIG. 10.

Let PN be perpendicular to OQ.

Then

$$\begin{aligned}\delta(\tan\theta) &= \text{PQ}\\ &= \text{PN}\sec\angle\text{NPQ}\\ &= \text{PN}\sec(\theta+\delta\theta)\\ &= \text{OP}\sec(\theta+\delta\theta)\sin\delta\theta\\ &= \sec\theta\sec(\theta+\delta\theta)\sin\delta\theta.\end{aligned}$$

Thus

$$\frac{\delta(\tan\theta)}{\delta\theta} = \sec\theta\sec(\theta+\delta\theta)\left(\frac{\sin\delta\theta}{\delta\theta}\right),$$

and proceeding to the limit,

$$\frac{d}{d\theta}\tan\theta = \sec^2\theta.$$

EXAMPLES. Find $\frac{dy}{dx}$ in the following cases :—

(i.) $y = 2a\sin(bx+c)\sin(bx-c)$.

(ii.) $y = x^2\cos 2x$.

(iii.) $y = \tan 3x + \cot 3x$.

(iv.) $y = \frac{\sin 2x - \sin x}{\cos x}$.

(v.) $y = x^m \sin^n x$.

(vi.) $y = x^m \sin nx$.

(vii.) $y = \sin^p x \cos^q x$.

(viii.) $y = \sec^2(ax+b) + \text{cosec}^2(cx+d)$.

§ 26. **The Graphs of the Trigometrical Functions.**

The results, which we can deduce from the differential coefficients of the functions

$$\sin x, \quad \cos x \quad \text{and} \quad \tan x,$$

should be compared with the information to be obtained from the graphs of these functions.

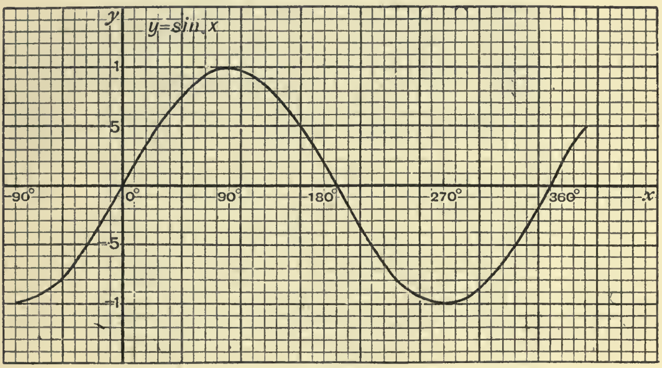
These graphs are given—when the angle is measured in degrees and with a suitable scale—in Figs. 11, 12 and 13.

It must be noticed that when x is the number of degrees *in the angle whose sine is y, the differential coefficient* $\frac{dy}{dx}$ *is not* $\cos x$.

It is a good exercise for the reader to show that in this case

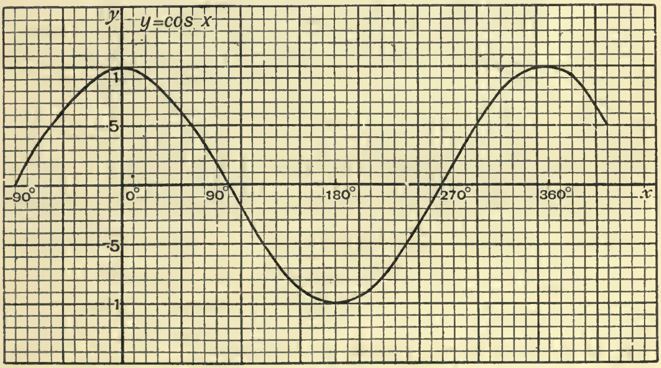
$$\frac{dy}{dx}=\frac{\pi}{180}\cos x.$$

A similar change has to be made in the differential coefficients of the other Trigonometrical Functions, when they are not measured in radians.



$y=\sin x.$

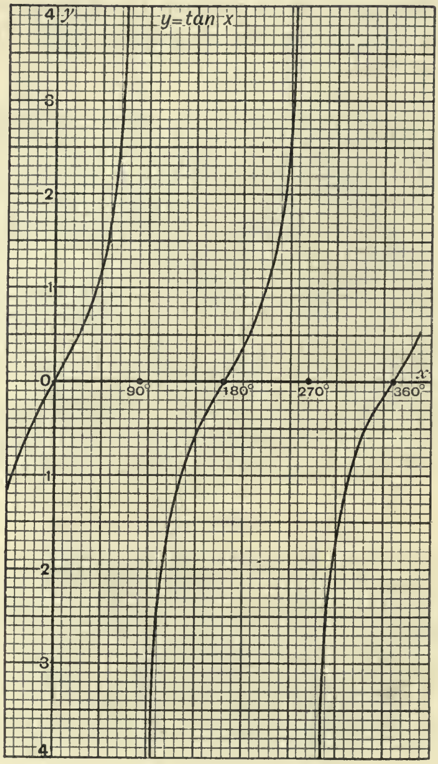
FIG. 11.



$y=\cos x.$

FIG. 12.

However, the general behaviour of the functions—when they are increasing, and when decreasing; when they reach their maxima or minima, if such exist; when their graphs are concave upwards, or convex upwards, etc.—can be seen from these figures.



$y = \tan x.$

FIG. 13.

Ex. In the Four-Figure Tables, we are told that

$$\sin 46^\circ \quad = \cdot 7193$$

$$\text{and} \qquad \sin 46^\circ\ 6' = \cdot 7206.$$

Compare this result with that obtained by the Calculus method.

Use $\qquad \cos 46^\circ = \cdot 6947.$

THE INVERSE TRIGONOMETRICAL FUNCTIONS

§ 27. The Differentiation of the Inverse Sine.

To any value of x, lying between -1 and $+1$, there corresponds an infinite number of angles which have this value x for their sine. If y is the circular measure of one of these angles, then

$$\sin y = x$$

is the equation connecting x and y.

If we give to y a number of values, we can obtain from the Tables the corresponding values of x, and in this way plot the curve

$$\sin y = x.$$

It is clear that it is a periodic curve of period 2π in y, and that it could be derived from the sine curve by placing this curve along the axis of y, instead of along the axis of x.

Another way of drawing the curve—and this is common to all such inverse curves—is to fold the paper, on which the curve

$$y = \sin x$$

is drawn about the line

$$y = x,$$

and this sine curve will then coincide with the curve

$$\sin y = x.$$

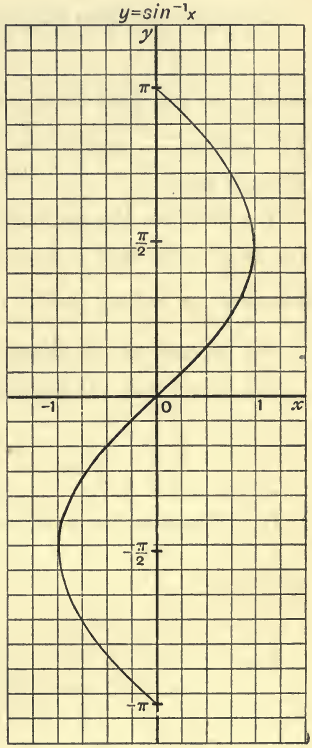


Fig. 14.

It is convenient to have a name and a symbol for this functional relation. If y is the circular measure of the angle whose sine is x, y is said to be the *inverse sine of* x, and the notation adopted is

$$y = \sin^{-1}x.$$

Part of the curve $\quad y = \sin^{-1}x$

is given in Fig. 14.

To save ambiguity and to make the function single-valued—that is to give only one value of y for one value of x—it is an advantage to restrict the symbol

$$\sin^{-1}x$$

to the number of radians in the angle between $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *whose sine is* x.

With this notation the curve

$$y = \sin^{-1}x$$

would be the part of the curve on Fig. 14 which lies between the values $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ of y. This is drawn on Fig. 14 in a heavier line.

We shall use the symbol $\sin^{-1}x$
for this value only: that is, *the angle whose sine is* x *is to be measured in radians and to lie between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$.

We proceed to the differentiation of $\sin^{-1}x$.

We begin with the equation

$$y = \sin^{-1}x. \quad \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$$

Then $$\sin y = x.$$

On differentiating both sides of this equation with regard to x,

we have $$\frac{d}{dx}\sin y = \frac{d}{dx}x = 1.$$

But $$\frac{d}{dx}\sin y = \frac{d}{dy}\sin y \times \frac{dy}{dx}$$

$$= \cos y \frac{dy}{dx}.$$

It follows that $$\cos y \frac{dy}{dx} = 1$$

and $$\frac{dy}{dx} = \frac{1}{\cos y}.$$

But we know that $\sin y = x$, and that $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Therefore we must have

$$\cos y = \sqrt{1-x^2},$$

the square root being taken with the positive sign.

Hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Therefore the differential coefficient of $\sin^{-1}x$ is $\frac{1}{\sqrt{1-x^2}}$.

It will be noticed that if we take the complete curve for the inverse sine, instead of the portion from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ only, the gradients at the points where $x = const.$ cuts the curve are alternately

$$\pm\frac{1}{\sqrt{1-x^2}}.$$

Ex. 1. **If $y = \sin^{-1}\frac{x}{a}$, $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-x^2}}$.**

We have $y = \sin^{-1}u$, where $u = \frac{x}{a}$.

But $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

It follows that $$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2-x^2}}.$$

2. If $y = \sin^{-1}(x^2)$, $\frac{dy}{dx} = \frac{2x}{\sqrt{1-x^4}}$.

We have $y = \sin^{-1}u$, where $u = x^2$.

But $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

It follows that $$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times 2x$$

$$= \frac{2x}{\sqrt{1-x^4}}.$$

3. If $y = \sin^{-1}\left(\frac{1-x}{\sqrt{2}}\right)$, $\frac{dy}{dx} = -\frac{1}{\sqrt{1+2x-x^2}}$.

4. Fill up the blanks in the following table:

$f(x)$.	$f'(x)$.
$\sin^{-1}(1+x)$	
$\sin^{-1}\sqrt{x}$	
$\sin^{-1}\dfrac{x}{2a}$	
$\sin^{-1}\sqrt{\dfrac{x}{2a}}$	
$\sin^{-1}\left(\dfrac{x-a}{a}\right)$	

§ 28. The Differentiation of the Inverse Cosine.

To any value of x, lying between -1 and $+1$, there corresponds an infinite number of angles which have this value x for their cosine. If y is the circular measure of one of these angles,

$$\cos y = x$$

is the equation connecting x and y.

This relation is also expressed by the notation

$$y = \cos^{-1}x,$$

and y is said to be *the inverse cosine of* x.

Part of the curve

$$y = \cos^{-1}x$$

is given in Fig. 15.

In the case of the inverse cosine it is again convenient to make the function single-valued. For this purpose it is best to restrict the symbol

$$\cos^{-1}x$$

to the number of radians in the angle between 0 *and* π *whose cosine is* x.

With this notation the curve

$$y = \cos^{-1}x$$

would be that part of the curve in Fig. 15, lying between the values 0 and π of y. It is drawn with a heavier line in that figure.

We proceed to the differentiation of the inverse cosine.

We begin with the equation

$$y = \cos^{-1} x. \quad (0 < y < \pi)$$

Then $$\cos y = x.$$

On differentiating both sides of this equation with regard to x, we have

$$\frac{d}{dx}\cos y = \frac{d}{dx}x = 1.$$

But $$\frac{d}{dx}\cos y = \frac{d}{dy}\cos y \times \frac{dy}{dx}$$

$$= -\sin y \frac{dy}{dx}.$$

It follows that

$$-\sin y \frac{dy}{dx} = 1.$$

and $$\frac{dy}{dx} = -\frac{1}{\sin y}.$$

But we know that $\cos y = x$ and that $0 < y < \pi$.

Therefore we must have

$$\sin y = \sqrt{1 - x^2},$$

the square root being taken with the positive sign.

Hence

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

Therefore the differential coefficient of $\cos^{-1}x$ is

$$-\frac{1}{\sqrt{1-x^2}}.$$

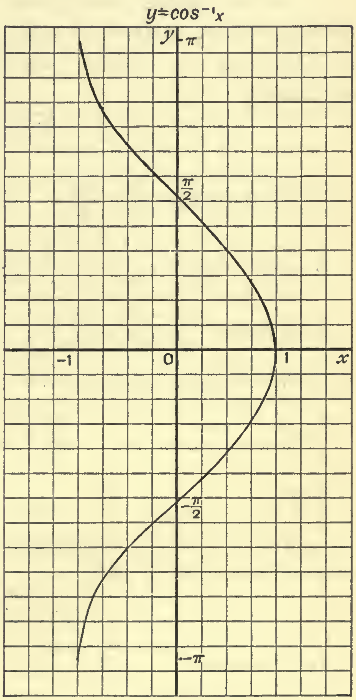


Fig. 15.

This result could have been deduced from § 27, since, with the notation we have adopted,

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

For example,
$$\left.\begin{aligned}\sin^{-1}\frac{1}{2}&=\frac{\pi}{6}\\ \cos^{-1}\frac{1}{2}&=\frac{\pi}{3},\end{aligned}\right\}$$
$$\left.\begin{aligned}\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)&=-\frac{\pi}{4}\\ \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)&=\frac{3\pi}{4}.\end{aligned}\right\}$$

It will be noticed that if we take the complete curve for the inverse cosine, instead of only the portion from 0 to π, the gradients at the points where $x=const.$ cuts the curve are alternately
$$\pm\frac{1}{\sqrt{1-x^2}}.$$

Ex. 1. If $y=\cos^{-1}\left(\frac{x}{a}\right)$, $\frac{dy}{dx}=-\frac{1}{\sqrt{a^2-x^2}}$.

2. If $y=\cos^{-1}(x^3)$, $\frac{dy}{dx}=-\frac{3x^2}{\sqrt{1-x^6}}$.

3. If $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $\frac{dy}{dx}=\frac{2}{1+x^2}$.

In this example, we have $y=\cos^{-1}u$, where $u=\frac{1-x^2}{1+x^2}$.

But
$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

Also
$$\frac{dy}{du}=-\frac{1}{\sqrt{1-u^2}}=-\frac{(1+x^2)}{2x}.$$

And
$$\frac{du}{dx}=\frac{(1+x^2)(-2x)-(1-x^2)2x}{(1+x^2)^2}$$
$$=-\frac{4x}{(1+x^2)^2}.$$

It follows that
$$\frac{dy}{dx}=\frac{2}{1+x^2}.$$

4. Fill up the blanks in the following table:

$f(x)$	$\cos^{-1}(1-x)$	$\cos^{-1}\left(\frac{x}{a}\right)^2$	$\cos^{-1}\sqrt{\frac{x}{a}}$	$\frac{1}{2}\cos^{-1}(2x-1)$	$x\cos^{-1}x$
$f'(x)$					

§ 29. **The Differentiation of the Inverse Tangent.**

To any value of x lying between $-\infty$ and ∞, there corresponds an infinite number of angles which have this value x for their tangent. If y is the circular measure of one of these angles,

$$\tan y = x$$

is the equation connecting x and y.

This relation is also expressed by the notation

$$y = \tan^{-1}x,$$

and y is said to be *the inverse tangent of x.*

Part of the curve $\qquad y = \tan^{-1}x$
is given in Fig. 16.

$y = tan^{-1}x$

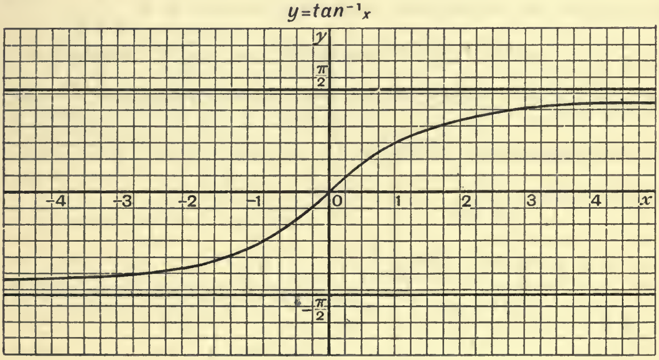


FIG. 16.

In the case of the inverse tangent it is again convenient to make the function single-valued, and this is done by restricting the symbol $\tan^{-1}x$ *to the number of radians in the angle between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *whose tangent is* x

With this notation the curve

$$y = \tan^{-1}x$$

would be the part of the complete curve of the inverse tangent for the values $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ of y. It is this part of the curve which is given in Fig. 15.

We proceed to the differentiation of the inverse tangent.

We begin with the equation

$$y = \tan^{-1}x. \quad \left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$$

Then $\tan y = x.$

On differentiating both sides of this equation with regard to x, we have

$$\frac{d}{dx}\tan y = \frac{d}{dx}x = 1.$$

But $$\frac{d}{dx}\tan y = \frac{d}{dy}\tan y \times \frac{dy}{dx} = \sec^2 y\frac{dy}{dx}.$$

And $$\sec^2 y = 1 + \tan^2 y = 1 + x^2.$$

Therefore $$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

Therefore the differential coefficient of $\tan^{-1}x$ is $\frac{1}{1+x^2}$.

It will be noticed that if we take the complete curve for the inverse tangent, instead of only the portion between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, the gradients at the points where $x = const.$ cuts the curve have the same value $\frac{1}{1+x^2}$.

Ex. 1. If $y = \tan^{-1}\frac{x}{a}$, $\frac{dy}{dx} = \frac{a}{a^2+x^2}$.

2. If $y = \tan^{-1}\left(\frac{1}{x}\right)$, $\frac{dy}{dx} = -\frac{1}{1+x^2}$.

We have $y = \tan^{-1}u$, where $u = \frac{1}{x}$.

But $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Also $$\frac{dy}{du} = \frac{1}{1+u^2} = \frac{x^2}{1+x^2}.$$

And $$\frac{du}{dx} = -\frac{1}{x^2}.$$

It follows that $$\frac{dy}{dx} = -\frac{1}{1+x^2}.$$

It will be noticed that the angle whose cotangent is x has $\frac{1}{x}$ for its tangent. **This example shows us that the differential coefficient of the inverse cotangent of x is $-\frac{1}{1+x^2}$.**

This result would also follow from the equation

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2},$$

which is true if we take $\cot^{-1}x$ to lie between 0 and π, while $\tan^{-1}x$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

3. If $y=\tan^{-1}(x^4)$, $\frac{dy}{dx}=\frac{4x^3}{1+x^8}$.

4. If $y=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$, $\frac{dy}{dx}=\frac{1}{x^2+x+1}$.

In this example, we have

$$y=\frac{2}{\sqrt{3}}\tan^{-1}u, \quad \text{where} \quad u=\frac{2x+1}{\sqrt{3}}.$$

But
$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}.$$

Working out $\frac{dy}{du}$, we find that it is equal to $\frac{\sqrt{3}}{2(x^2+x+1)}$.

And
$$\frac{du}{dx}=\frac{2}{\sqrt{3}}.$$

It follows that
$$\frac{dy}{dx}=\frac{1}{x^2+x+1}.$$

5. If $y=x\tan^{-1}x$, $\frac{dy}{dx}=\frac{x}{1+x^2}+\tan^{-1}x$.

Using the rule for differentiating the product of two functions, we have

$$\frac{dy}{dx}=x\frac{d}{dx}\tan^{-1}x+\tan^{-1}x\frac{d}{dx}x$$

$$=\frac{x}{1+x^2}+\tan^{-1}x.$$

6. Fill up the blanks in the following table:

$f(x)$	$\tan^{-1}(2x-1)$	$\tan^{-1}\left(\frac{1}{x^2}\right)$	$\tan^{-1}\frac{1}{\sqrt{x}}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$x^2\tan^{-1}x^2$
$f'(x)$					

EXAMPLES ON CHAPTER IV

1. Differentiate the following functions :—

(i.) $\sin^3x+\cos^3x$.

(ii.) $\tan x+\frac{1}{3}\tan^3x$.

(iii.) $\sec^2x+\tan^2x$.

(iv.) $\operatorname{cosec}^2x+\cot^2x$.

(v.) $\frac{1+\sin x}{1-\sin x}$.

(vi.) $\frac{1-\cos x}{1+\cos x}$.

2. If $y=\dfrac{\sin x}{1+\tan x}$, prove that $\dfrac{dy}{dx}=\dfrac{\cos^3 x-\sin^3 x}{(\cos x+\sin x)^2}$.

3. If $y=\cos(x^3)$, prove that $\dfrac{dy}{dx}=-3x^2\sin(x^3)$, and find $\dfrac{dy}{dx}$ when

(i.) $y=x^m \sin x^n$.
(ii.) $y=x^m \cos x^n$.
(iii) $y=x^m \tan x^n$.

4. Differentiate the following functions :—

(i.) $(x^2+1)\tan^{-1}x-x$.
(ii.) $x\sin^{-1}x+\sqrt{1-x^2}$.
(iii.) $\tan^{-1}\left(\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}\right)$. (Put $\sqrt{x}=\tan\theta$, $\sqrt{a}=\tan\alpha$.)
(iv.) $\tan^{-1}\left(\dfrac{1+x+x^2}{1-x+x^2}\right)$.
(v.) $\cot^{-1}\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$. (Put $x=\tan\theta$.)

5. A particle P is revolving with constant angular velocity ω in a circle of radius a. The line PM is drawn from P perpendicular to the line from the centre to the initial position of the particle. Find the velocity and acceleration of M.

6. If the position of a point is given at time t by the equations

$$x=a(\omega t+\sin\omega t),$$
$$y=a(1-\cos\omega t),$$

where a and ω are constants, find its component velocities and accelerations, and its direction of motion at the time t.

7. Prove that when

$$x<-1,\quad \frac{d}{dx}(\sec^{-1}x)=-\frac{1}{x\sqrt{x^2-1}},$$

and that when $$x>1,\quad \frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}},$$

and illustrate your results from the graph of the inverse secant.

8. Prove that when

$$x>1,\quad \frac{d}{dx}(\operatorname{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}},$$

and that when $$x<-1,\quad \frac{d}{dx}(\operatorname{cosec}^{-1}x)=\frac{1}{x\sqrt{x^2-1}},$$

and illustrate your results from the graph of the inverse cosecant.

CHAPTER V

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS—MAXIMA AND MINIMA—PARTIAL DIFFERENTIATION

§ 30. Introductory.

There remain two important functions which we must learn to differentiate: the logarithm of x to the base a, and its inverse function a^x.

We shall find that there is a particular number denoted by e for which the Logarithmic Function

$$\log_e x$$

and the Exponential Function

$$e^x$$

are of the greatest importance.

The differential coefficients which we require can be obtained much more quickly with the aid of Infinite Series, and those who are familiar with that branch of Algebra will probably prefer the usual method of finding them given in the Appendix. In the articles which follow we obtain them without using more than Elementary Algebra and the Logarithm Tables. It is true that this discussion, in one or two points, is not quite rigorous. Still those for whom the rigorous treatment is suitable will get it in their later course. Those who do not carry their study of the Calculus further will yet have obtained a working knowledge of the meaning of the new functions and a complete enough grasp of the application of the Calculus to them.

The following formulae in logarithms are supposed known:

$$\log \mathrm{MN} = \log \mathrm{M} + \log \mathrm{N}, \qquad (1)$$

$$\log \frac{\mathrm{M}}{\mathrm{N}} = \log \mathrm{M} - \log \mathrm{N}, \qquad (2)$$

$$\log \mathrm{M}^n = n \log \mathrm{M}. \qquad (3)$$

These are true for any base. All the numbers are supposed to be positive.

By a simple application of the Index Laws, another formula is obtained, which allows us to change logarithms from one base to another. This formula is

$$\log_a N = \frac{\log_b N}{\log_b a}. \qquad (4)$$

If we put $N = b$ in this equation, we have

$$\log_a b \times \log_b a = 1. \qquad (5)$$

Thus we can write (4) in the form

$$\log_a N = \log_a b \times \log_b N. \qquad (6)$$

Since $$\log_a x = \log_a b \log_b x,$$

we have $$\frac{d}{dx}\log_a x = \log_a b \frac{d}{dx}\log_b x.$$

It follows that, if we know the differential coefficient of the logarithm of x to any base a, we can write down that for any other base b, it being of course understood that the bases a and b are independent of x.

In the discussion which follows we shall first find

$$\frac{d}{dx}\log_{10} x;$$

but, before we can do so, it will be necessary to learn something about the behaviour of the expression

$$n \log_{10}\left(1 + \frac{1}{n}\right),$$

as n gets larger and larger.

In the work on Algebra, which we are omitting, it is proved rigorously that the number $\left(1 + \frac{1}{n}\right)^n$ continually increases as n increases, and that when $n \to \infty$, it has a definite limiting value.

In other words, $$\underset{n \to \infty}{\mathrm{Lt}} \left(1 + \frac{1}{n}\right)^n$$

is a definite number.

It is true that this number is incommensurable, but its value can be obtained to as close a degree of accuracy as is required. Correct to 7 places of decimals it is 2·7182818.

This number is denoted by e. It is the base of the Napierian or natural system of logarithms.

From the result that

$$\underset{n \to \infty}{\mathrm{Lt}} \left(1 + \frac{1}{n}\right)^n = e,$$

it follows that

$$\operatorname*{Lt}_{n \to \infty} n \log_a\left(1 + \frac{1}{n}\right) = \log_a e,$$

where a is any base.

In particular, if we take the base 10 and look up the logarithm of 2·7182818 in the Tables, we find that

$$\operatorname*{Lt}_{n \to \infty} n \log_{10}\left(1 + \frac{1}{n}\right) = \cdot 4342945.$$

Without assuming the truth of any of the above work, we shall now see what information the Tables give us regarding the expression

$$n \log_{10}\left(1 + \frac{1}{n}\right).$$

§ 31. **The Expressions**

$$\left(1 + \frac{1}{n}\right)^n \text{ and } n \log_{10}\left(1 + \frac{1}{n}\right).$$

In the accompanying tables the approximate values of

$$n \log_{10}\left(1 + \frac{1}{n}\right)$$

are given for $n = 1$, 50, 100, 500, etc. The figures in Column II. are calculated from 7-Figure Logarithm Tables; those in Column III. from 8-Figure Tables.

TABLE

Showing the value of $n \log_{10}\left(1 + \frac{1}{n}\right)$.

n	7-Figure Tables.	8-Figure Tables.
1	·3 0 1 0 3 0 0	·3 0 1 0 3 0 0 0
50	·4 3 0 0 1	·4 3 0 0 0 9
100	·4 3 2 1 4	·4 3 2 1 3 7
500	·4 3 3 8	·4 3 3 8 6 5
1000	·4 3 4 1	·4 3 4 0 8
2000	·4 3 4 2	·4 3 4 1 8
3000	·4 3 4 1	·4 3 4 2 5
4000	·4 3 4 4	·4 3 4 2 4
5000	·4 3 4 5	·4 3 4 2 5
6000	·4 3 3 8	·4 3 4 2 8
7000	·4 3 4 7	·4 3 4 2 8
8000	·4 3 4 4	·4 3 4 2 4
9000	·4 3 4 7	·4 3 4 2 5
10000	·4 3 4	·4 3 4 3

It will be seen that the values we have obtained for

$$n \log_{10}\left(1+\frac{1}{n}\right)$$

increase with n until we reach $n=2000$ in the second column, and $n=3000$ in the third.

The oscillation that we meet there, and in some of the later numbers, is due to the fact that in 7-Figure Logarithm Tables the seventh decimal place is only the *nearest* value, and may err to the extent of ·5 either way. When the logarithm is multiplied by 1000, the unknown error in the product comes within ·5 either way of the fourth decimal place. In the products from 2000 to 9000 this may affect the fourth decimal, and even the third.

The same argument applies to the results in Column III. from 8-Figure Tables, and in this way the oscillations, when $n=3000$ and 4000, and when $n=7000$ and 8000, can be explained.

To avoid this source of error, and to show still more clearly the behaviour of the expression

$$n \log_{10}\left(1+\frac{1}{n}\right)$$

as n increases, the following table has been calculated, *correct to ten places,* using 15-Figure Logarithm Tables.

TABLE

Showing value of $n \log_{10}\left(1+\frac{1}{n}\right)$ correct to 10 places.

n	$n \log_{10}\left(1+\frac{1}{n}\right)$.
1	·3 0 1 0 2 9 9 9 5 7
10	·4 1 3 9 2 6 8 5 1 6
25	·4 2 5 8 3 3 4 8 2 5
50	·4 3 0 0 0 8 5 8 8 1
100	·4 3 2 1 3 7 3 7 8 3
500	·4 3 3 8 6 0 7 6 5 6
1000	·4 3 4 0 7 7 4 7 9 3
10000	·4 3 4 2 7 2 7 6 8 6
100000	·4 3 4 2 9 4 2 6 4 8

For $n=1{,}000{,}000$, we find $n \log_{10}\left(1+\frac{1}{n}\right)=·434294460$.

It will be seen from these results that we may safely assume that as n gets larger and larger

$$n \log_{10}\left(1+\frac{1}{n}\right)$$

gets very near the number 0·4343; and we find from the Tables that

$$\log_{10} 2{\cdot}718 = {\cdot}4343.$$

We shall therefore assume that

$$\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n$$

exists, and we shall denote it by e.

We shall take as our approximation to e the number 2·718, and we shall take for our approximation to $\log_{10}$e the number ·4343.

We are now able to proceed to the differentiation of the logarithm of x to any base. We shall begin with the base 10, and then find the differential coefficients of $\log_e x$ and $\log_a x$.

From these results we shall readily obtain the differential coefficients of e^x and a^x.

§ 32. **The Differentiation of $\log_{10}$x.** (Cf. App. p. 130.)

We begin with the value x, and we put

$$y = \log_{10} x.$$

Then we take the increment δx of x, and we write δy for the corresponding increment of y.

It follows that $\quad y + \delta y = \log_{10}(x+\delta x).$

Therefore we have

$$\delta y = \log_{10}(x+\delta x) - \log_{10} x$$

$$= \log_{10}\left(1+\frac{\delta x}{x}\right).$$

Therefore $\quad \dfrac{\delta y}{\delta x} = \dfrac{1}{\delta x}\log_{10}\left(1+\dfrac{\delta x}{x}\right)$

$$= \frac{1}{x}\times\frac{x}{\delta x}\log_{10}\left(1+\frac{\delta x}{x}\right).$$

Now put $n = \dfrac{x}{\delta x}$ on the right-hand side.

Remembering that x is fixed, we see that when the increment of x, namely δx, is made smaller and smaller, n gets larger and larger.

Also as $\delta x \to 0$, $n \to \infty$.

Therefore we have

$$\underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\delta y}{\delta x}\right) = \frac{1}{x} \times \underset{n \to \infty}{\mathrm{Lt}} \left\{ n \log_{10}\left(1 + \frac{1}{n}\right)\right\}.$$

It follows from § 31, that

$$\frac{dy}{dx} = \frac{\log_{10} e}{x}.$$

Thus the differential coefficient of $\log_{10} x$ is $\frac{\log_{10} e}{x}$.

§ 33. **The Differentiation of $\log_e x$ and $\log_a x$.** (Cf. App. p. 130.)

Since $$\log_e x = \frac{\log_{10} x}{\log_{10} e}, \quad \text{[Cf. § 30 (4)]}$$

it follows that $$\frac{d}{dx} \log_e x = \frac{1}{\log_{10} e} \frac{d}{dx} \log_{10} x$$

$$= \frac{1}{\log_{10} e} \times \frac{1}{x} \log_{10} e.$$

Therefore $$\frac{d}{dx} \log_e x = \frac{1}{x}.$$

Again we have $$\log_a x = \frac{\log_e x}{\log_e a}. \quad \text{[Cf. § 30 (4)]}$$

It follows that $$\frac{d}{dx} \log_a x = \frac{1}{\log_e a} \frac{d}{dx} \log_e x$$

$$= \frac{1}{\log_e a} \times \frac{1}{x}.$$

Therefore $$\frac{d}{dx} \log_a x = \frac{1}{x \log_e a} = \frac{\log_a e}{x}. \quad \text{[Cf. § 30 (5)]}$$

In Elementary Trigonometry it is convenient to write $\log N$ for $\log_{10} N$. In the Calculus and in Higher Mathematics we usually write $\log N$ for $\log_e N$: that is, we only insert the base of the logarithm $$\log_a x$$
when the base a is different from e.

However sometimes we shall insert the base e, if it is necessary to emphasise the fact that logarithms are being taken to that base. With this notation the results of this section are written

$$\frac{d}{dx}(\log x)=\frac{1}{x}, \qquad (x>0)$$

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\log a}=\frac{\log_a e}{x} \qquad (x>0)$$

The equation $\frac{d}{dx}\log x=\frac{1}{x}$ *is of the greatest possible importance.*

Ex. 1. If $y=\log(ax+b)$, $\frac{dy}{dx}=\frac{a}{ax+b}$.

We have $y=\log u$, where $u=ax+b$.

But $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$.

Therefore $\frac{dy}{dx}=\frac{1}{u}\times a=\frac{a}{ax+b}$.

2. If $y=\log(ax^2+2bx+c)$, $\frac{dy}{dx}=\frac{2(ax+b)}{ax^2+2bx+c}$.

3. If $y=\log\sin x$, $\frac{dy}{dx}=\cot x$.

4. If $y=\log\cos x$, $\frac{dy}{dx}=-\tan x$.

5. If $y=\log\tan\frac{x}{2}$, $\frac{dy}{dx}=\frac{1}{\sin x}$.

We have $y=\log u$, where $u=\tan\frac{x}{2}$.

It follows that

$$\frac{dy}{dx}=\frac{1}{u}\times\frac{1}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\frac{1}{\sin x}.$$

6. **If $y=\log f(x)$,** $\frac{dy}{dx}=\frac{f'(x)}{f(x)}$.

7. Fill up the blanks in the following table :—

$f(x)$	$\log(1-x)$	$\log(1+x)$	$\log\left(\frac{1-x}{1+x}\right)$	$\log(x-a)$	$\log(x+a)$	$\log\left(\frac{x-a}{x+a}\right)$	$\log\left(\frac{a-x}{a+x}\right)$
$f'(x)$							

8. If $y=\log\dfrac{a^2-x^2}{b^2-x^2}$, $\qquad \dfrac{dy}{dx}=2x\dfrac{a^2-b^2}{(a^2-x^2)(b^2-x^2)}$.

We have $\qquad y=\log(a^2-x^2)-\log(b^2-x^2)$.

Therefore
$$\frac{dy}{dx}=-\frac{2x}{a^2-x^2}+\frac{2x}{b^2-x^2}$$
$$=2x\frac{a^2-b^2}{(a^2-x^2)(b^2-x^2)}.$$

9. If $y=\log(x\pm\sqrt{x^2\pm a^2})$, $\qquad \dfrac{dy}{dx}=\dfrac{1}{\sqrt{x^2\pm a^2}}$.

10. If $y=\log\sqrt{\dfrac{a-b\cos x}{a+b\cos x}}$, $\qquad \dfrac{dy}{dx}=\dfrac{ab\sin x}{a^2-b^2\cos^2 x}$.

§ 34. **The Differentiation of e^x.** (Cf. App. p. 130.)

Let $\qquad y=e^x$.

Then we have $\qquad \log y=x\log e$.

Therefore $\qquad \log y=x$, since $\log_e e=1$.

Differentiating both sides of this equation with regard to x, we have
$$\frac{d}{dx}\log y=1.$$

But
$$\frac{d}{dx}\log y=\frac{d}{dy}\log y\times\frac{dy}{dx}$$
$$=\frac{1}{y}\frac{dy}{dx}.$$

It follows that
$$\frac{1}{y}\frac{dy}{dx}=1.$$

Therefore
$$\frac{dy}{dx}=y=e^x.$$

Thus the differential coefficient of e^x is e^x.

The equation $\dfrac{d}{dx}e^x=e^x$ *is of the greatest possible importance.*

Ex. 1. If $y=e^{mx}$, $\qquad \dfrac{dy}{dx}=me^{mx}$.

We have $\qquad y=e^u$, where $u=mx$.

It follows that
$$\frac{dy}{dx}=e^u\times m$$
$$=me^{mx}.$$

2. If $y=e^{ax+b}$, $\qquad \dfrac{dy}{dx}=ae^{ax+b}$.

3. If $y=e^{ax}\sin bx$, $\qquad \dfrac{dy}{dx}=e^{ax}[a\sin bx+b\cos bx]$.

4. If $y=e^{-ax}\sin bx$, $\quad \dfrac{dy}{dx}=e^{-ax}[b\cos bx-a\sin bx]$.

5. If $y=ae^{bx}$, $\quad \dfrac{dy}{dx}=by$.

6. If $y=e^{x^2}$, $\quad \dfrac{dy}{dx}=2xy$.

7. If $y=xe^{x^2}$, $\quad x\dfrac{dy}{dx}=(1+2x^2)y$.

§ 35. **The Differentiation of $\mathbf{a^x}$.**

Let $\quad y=a^x$.

Also let $\quad \log_e a=m$.

Then $\quad e^m=a$.

Therefore we have $\quad y=e^{mx}$.

It follows that $\quad \dfrac{dy}{dx}=me^{mx}$.

Therefore $\quad \mathbf{\dfrac{d}{dx}a^x=a^x\log_e a}$.

It must be noticed that the number a in this formula is a constant and independent of x.

Ex. Prove that the differential coefficient of a^x is $a^x\log a$, by taking logarithms of both sides of the equation

$$y=a^x$$

and then differentiating.

§ 36. **Logarithmic Differentiation.**

We have already obtained a general rule for the differentiation of a product or quotient. We are now able to prove another method which often leads more quickly to the result. This method is called *Logarithmic Differentiation.*

Let $\quad y=uvw$.

Then $\quad \log y=\log u+\log v+\log w$.

$$\therefore\ \frac{d}{dx}(\log y)=\frac{d}{dx}(\log u)+\frac{d}{dx}(\log v)+\frac{d}{dx}(\log w).$$

$$\therefore\ \frac{d}{dy}(\log y)\frac{dy}{dx}=\frac{d}{du}(\log u)\frac{du}{dx}+\frac{d}{dv}(\log v)\frac{dv}{dx}+\frac{d}{dw}(\log w)\frac{dw}{dx}.$$

$$\therefore\ \frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}.$$

Multiplying up by y, we have the value of $\dfrac{dy}{dx}$.

In other words, before differentiating an equation involving the product or quotient or powers of other expressions, take logarithms of both sides of the given equation.

Thus when we are given an equation involving the product, or quotient, or powers of several expressions, it is often an advantage to take logarithms of both sides of the given equation before differentiating.

Ex. 1. If $y=\sqrt{\dfrac{1-x^2}{1+x^2}}$, $\quad \dfrac{dy}{dx}=-\dfrac{2x}{\sqrt{(1-x^2)(1+x^2)^3}}$.

We have $$\log y=\frac{1}{2}\log(1-x^2)-\frac{1}{2}\log(1+x^2).$$

Therefore $$\frac{1}{y}\frac{dy}{dx}=-\frac{x}{1-x^2}-\frac{x}{1+x^2}$$
$$=-\frac{2x}{1-x^4}.$$

Hence $$\frac{dy}{dx}=-\frac{2x}{\sqrt{(1-x^2)(1+x^2)^3}}.$$

2. If $y=(1-x)^3(1-2x)^4$, find $\dfrac{dy}{dx}$.

We have $$\log y=3\log(1-x)+4\log(1-2x).$$

Therefore $$\frac{1}{y}\frac{dy}{dx}=-\frac{3}{1-x}-\frac{8}{1-2x}$$
$$=-\frac{(11-14x)}{(1-x)(1-2x)}.$$

Hence $$\frac{dy}{dx}=-(11-14x)(1-x)^2(1-2x)^3.$$

3. If $y=\dfrac{(ax+b)^p(cx+d)^q}{(ex+f)^s}$, $\quad \dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{ap}{ax+b}+\dfrac{qc}{cx+d}-\dfrac{se}{ex+f}$.

4. If $y=\sqrt{\dfrac{1+x^4}{1-x^4}}$, $\quad \dfrac{dy}{dx}=\dfrac{4x^3}{\sqrt{(1+x^4)(1-x^4)^3}}$.

5. If $y=\sqrt{\dfrac{a+2bx+cx^2}{a-2bx+cx^2}}$, $\quad \dfrac{dy}{dx}=\dfrac{b(a-cx^2)}{(a-2bx+cx^2)^{\frac{3}{2}}(a+2bx+cx^2)^{\frac{1}{2}}}$.

§ 37. Important Example.

If $$y=e^{-ax}\sin bx,$$
$$\frac{dy}{dx}=\sin bx\frac{d}{dx}(e^{-ax})+e^{-ax}\frac{d}{dx}(\sin bx)$$
$$=e^{-ax}(-a\sin bx+b\cos bx).$$

Now if $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$, a and b being positive,

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

Therefore $\frac{dy}{dx} = -\sqrt{a^2+b^2}\, e^{-ax}(\sin bx\cos\alpha - \cos bx\sin\alpha)$

$$= -\sqrt{a^2+b^2}\, e^{-ax}\sin(bx-\alpha).$$

Thus the tangent to the curve $y = e^{-ax}\sin bx$ is parallel to the axis of x, when

$$bx = n\pi + \alpha,$$

and the equation defines an oscillating curve with continually diminishing amplitude in the waves as we proceed along Ox.

It is easy to show that when

$$y = e^{ax}\sin(bx+c),$$

$$\frac{dy}{dx} = \sqrt{a^2+b^2}\, e^{ax}\sin(bx+c+\alpha),$$

and that here the waves increase in amplitude. Corresponding results hold for the case of the cosine.

§ 38. **Maxima and Minima Values of a Function of one Variable.**

The student is already familiar with the graphical and algebraical discussion of the maxima and minima of certain simple algebraical expressions. The methods of the Differential Calculus are well adapted to the solution of such problems.

If the graph of the function is supposed drawn, the *turning-points*, or places where the ordinate changes from increasing to decreasing, or *vice versa*, can only occur where the tangent is parallel to the axis of x, as in the points $A_1, A_2 \ldots$ of Fig. 17, or where it is parallel to the axis of y as in the points $B_1, B_2 \ldots$, except in such cases as the points $C_1, C_2 \ldots$, where, although the curve is continuous, the gradient suddenly changes sign, without passing through the value zero or becoming infinitely great.

In case (A): $\frac{dy}{dx}$ is zero at the turning-point; and if this point is one at which the curve ceases to ascend and begins to descend, $\frac{dy}{dx}$ changes from being positive just before that point to being negative just after. At such a point the function is said to have a *maximum* value. In the other case, where the curve ceases to descend and begins to ascend, $\frac{dy}{dx}$ changes from negative to positive, and we have a *minimum*. In Fig. 17, at A_1 there is a *maximum*; at A_2 there is a *minimum*.

In case B: $\frac{dy}{dx}$ is infinitely great at the turning-point; and at B_1, where there is a *maximum*, it changes from positive to

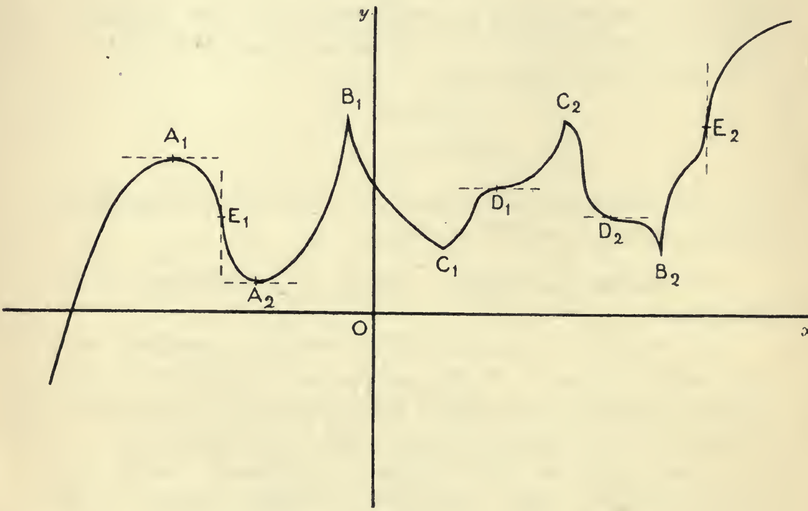


FIG. 17.

negative, while at B_2, where there is a *minimum*, it changes from negative to positive.

The other turning-points, C_1, C_2 in Fig. 17 correspond to discontinuities in $\frac{dy}{dx}$, but it can be shown that these will not occur in the functions with which we are dealing.

We thus obtain the following rule for finding the maxima and minima of a function $f(x)$, omitting cases B and C.

Obtain $f'(x)$ *and solve the equation* $f'(x) = 0$. *Let its roots be* x_1, x_2, ... *Examine the behaviour of* $f'(x)$ *in the neighbourhood of each of these roots.*

If $f'(x)$ changes from positive to negative as we pass through one of these roots, then $f(x)$ has a maximum value there.

If $f'(x)$ changes from negative to positive as we pass through one of these roots, then $f(x)$ has a minimum there.

If $f'(x)$ does not change sign as we pass through the root considered, then $f(x)$ has neither a maximum or a minimum there.

§ 39. **Points of Inflection.**

Although the vanishing of $\frac{dy}{dx}$ is a *necessary* condition for a maximum or minimum, it is not a *sufficient* condition, since the gradient of the curve may become zero without changing its sign as we pass through that point. Examples of such points are to be found in D_1, D_2 of Fig. 17. In the case of D_1, the gradient is positive before and after the zero value; in the case of D_2, it is negative. At these points the curve *crosses* its *tangent*, and when this occurs, whether the tangent is horizontal or not, the point is called a *point of inflection.*

From what we have already seen [cf. § 18] as to the conclusions we can draw from the sign of $\frac{d^2y}{dx^2}$, it is clear that, as we pass through the point D_1 of Fig. 17, $\frac{d^2y}{dx^2}$ changes from being negative to being positive. The curve is convex upwards just before D_1: it is concave upwards just after D_1. At D_1, $\frac{d^2y}{dx^2}=0$.

It will be seen on drawing a figure that at points where a curve crosses its tangent, the second differential coefficient *vanishes and changes sign*, provided that the gradient of the curve is continuous.

It is also easy to show that when $\frac{dy}{dx}=0$, and $\frac{d^2y}{dx^2}$ is negative, there is a maximum.

And when $\frac{dy}{dx}=0$, and $\frac{d^2y}{dx^2}$ is positive, there is a minimum.

Ex. 1. Show that $y=ax^2+2bx+c$ has always *one* turning-point; and point out when it is a maximum and when it is a minimum.

2. Find the maximum and minimum ordinates of the curve

$$y=x^3-6x^2+12,$$

and also find the points of maximum gradient.

3. Find the turning-points of the curve $y=(x+1)^3(x-2)^4$, and show that $(-1, 0)$ is a point of inflection.

4. Find the turning-points of $y=\frac{(x-1)}{(x-2)}$ and of $y=\frac{ax+b}{cx+d}$.

§ 40. Partial Differentiation.

So far we have been considering functions of only one independent variable, $y=f(x)$. Cases occur in Geometry and in all the applications of the Calculus where the quantities which vary depend upon more than one variable. For instance, in Geometry the co-ordinates of any point (x, y, z) upon the sphere of radius a, whose centre is at the origin, satisfy the relation

$$x^2+y^2+z^2=a^2.$$

Hence we have
$$z^2=a^2-x^2-y^2,$$

and if we cut the sphere by a plane parallel to the yz plane, along the circle where this plane cuts the sphere x is constant, and the change in z is due to a change in y only. In the section by a plane parallel to the zx plane, the change in z would be due to a change in x only. Similar results hold for other surfaces.

Again, the area of a rectangle whose sides are x in. and y in. is xy sq. in., and we may imagine the sides x and y to change in length independently of each other; while the volume of a rectangular box whose edges are x, y, and z in. is xyz cub. in., and x, y, z may be supposed to change independently.

The ordinary gas equation

$$\frac{pv}{\mathrm{T}}=\text{constant}$$

is another example of the same sort of relation, and it would be easy to multiply these instances indefinitely.

Let the equation
$$z=f(x, y)$$
express such a relation between two independent variables x and y, and a dependent variable z.

Let us suppose that the independent variable y is kept constant and that x changes.

Then the rate at which z changes with regard to x, when y is kept constant, will be given by

$$\underset{\delta x \to 0}{\mathrm{Lt}} \left\{\frac{f(x+\delta x, y)-f(x, y)}{\delta x}\right\}.$$

In the second case let x be kept constant and let y change. Then the rate at which z changes will be

$$\underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \right\}.$$

These two differential coefficients are called the Partial Differential Coefficients of z with regard to x and y respectively, and are written $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ respectively.*

Ex. 1. When $z=xy$, prove, from the definition, that $\frac{\partial z}{\partial x}=y$, and $\frac{\partial z}{\partial y}=x$.

2. When $2az=x^2+y^2$, prove, from the definition, that $\frac{\partial z}{\partial x}=\frac{x}{a}$, and $\frac{\partial z}{\partial y}=\frac{y}{a}$.

3. If $u=xyz$, prove, from the definition, that $\frac{\partial u}{\partial x}=yz$.

§ 41. Total Differentiation.

When the variables x and y in the above examples both depend upon a third variable t, z will vary in value, as x and y change with t.

In Ex. 1 above, $$z=xy,$$

$$z+\delta z=(x+\delta x)(y+\delta y).$$

Thus $$\frac{\delta z}{\delta t}=y\frac{\delta x}{\delta t}+x\frac{\delta y}{\delta t}+\frac{\delta x}{\delta t}\delta y.$$

Proceeding to the limit, we have

$$\frac{dz}{dt}=y\frac{dx}{dt}+x\frac{dy}{dt}.$$

But $$y=\frac{\partial z}{\partial x} \text{ and } x=\frac{\partial z}{\partial y}, \text{ when } z=xy.$$

Therefore, in this case,

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

In Ex. 2 above, $$2az=x^2+y^2,$$

we find $$2a\delta z=2x\delta x+2y\delta y+(\delta x)^2+(\delta y)^2.$$

---

*It is hardly necessary to point out that this symbol $\frac{\partial z}{\partial x}$ stands for an operation, and that ∂z, ∂x are not to be considered separately; also that this is a different notation from the δx of our earlier work.

Thus
$$a\frac{dz}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt};$$

so that again
$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

It can be shown that this holds in general, but the proof of the theorem cannot be taken at this stage of our work.

The differential coefficient $\frac{dz}{dt}$ is called the Total Differential Coefficient in such cases, as compared with the Partial Differential Coefficient defined above.

As a special case, when $z=f(x, y)$ and y is a function of x, we obtain
$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx},$$

and the left-hand side is called the Total Differential Coefficient of z with regard to x.

Also the result that, when $z=f(x, y)$ and x, y are functions of t,
$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

may be used to obtain an approximation to the small change δz in z due to the small changes δx and δy in x and y, when t becomes $t+\delta t$.

For, as we have already seen (p. 23),

δx will be approximately $\frac{dx}{dt}\delta t$;

δy will be approximately $\frac{dy}{dt}\delta t$;

and δz will be approximately $\frac{dz}{dt}\delta t$.

We thus have, on multiplying the above equation by δt,
$$\delta z=\frac{\partial z}{\partial x}\delta x+\frac{\partial z}{\partial y}\delta y, \text{ approximately.}$$

§ 42. Differentials.*

In the case of the curve $y=f(x)$, the increment δy of y which corresponds to the increment δx of x, is given in Fig. 18 by HQ.

* § 42 may be omitted on first reading.

Also $$HQ = HT + TQ = \delta x \frac{dy}{dx} + TQ.$$

$$\therefore \quad \delta y = \delta x \frac{dy}{dx} + TQ.$$

As δx gets smaller and smaller, TQ gets smaller and smaller, at least in the neighbourhood of P.

The "small quantity" TQ is a smaller "small quantity" than δx, since

$$\frac{\delta y}{\delta x} = \frac{dy}{dx} + \frac{TQ}{\delta x},$$

and in the limit $\frac{\delta y}{\delta x}$ is equal to $\frac{dy}{dx}$, so that $\frac{TQ}{\delta x}$ must disappear in the limit.

In mathematical language, if δx is an infinitesimal (or small quantity) of the first order, TQ will be at least an infinitesimal of the second order.

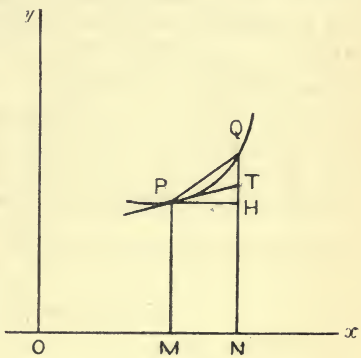


FIG. 18.

It is convenient to have a name and symbol for this quantity $\frac{dy}{dx}\delta x$. The name adopted is the "differential of y," and the symbol is "dy."

Hence with this definition of the term "differential,"

$$dy = \left(\frac{dy}{dx}\right)\delta x,$$

where we have enclosed $\frac{dy}{dx}$ in brackets on the right-hand side, so that it may be clear that this stands for the differential coefficient obtained by the processes we have been developing in the preceding pages.

By the above definition

$$d(f(x)) = f'(x)\,\delta x, \quad \text{where} \quad f'(x) = \frac{dy}{dx};$$

and $$dx = \delta x.$$

So that $$dy = f'(x)\,dx, \quad \text{when} \quad y = f(x).$$

Hence we may restate our definition as follows :—

The differential of the independent variable is the actual increment of that variable.

The differential of a function is the differential coefficient of the function multiplied by the differential of the independent variable.

In this definition it is not necessary to assume that the differentials are small quantities or infinitesimals, but in all the applications of this notation this assumption is made.

Then the equation $$dy = f'(x)\,dx$$
will give the increment of y, if small quantities of the second order be neglected.

Such an equation as $$dy = f'(x)\,dx,$$
a differential equation as it is called, may be used to give the approximate change in the dependent variable, and from this point of view it saves the trouble of writing down the equation between the increments, and then cutting out the terms which are so small that they may be neglected.

Ex. 1. Write down a table of differentials corresponding to the standard differential coefficients.

$$e.g.\ d(x^n) = nx^{n-1}dx.$$

2. If $x = a\cos\theta$, $y = a\sin\theta$, prove by differentials that $\frac{dy}{dx} = -\cot\theta$.

3. If $x = a(\omega t + \sin\omega t)$, $y = a(1-\cos\omega t)$, prove that $\frac{dy}{dx} = \frac{\sin\omega t}{1+\cos\omega t}$.

4. If $z = xy$, prove that $dz = \frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial y}dy$.

EXAMPLES ON CHAPTER V

1. Find the differential coefficients of

$$\text{(i.) } xe^x, \quad \text{(ii.) } x^m e^{nx}, \quad \text{(iii.) } (ax+b)e^{cx+d}, \quad \text{(iv.) } e^{x\sin^{-1}x}.$$

2. Find the differential coefficients of

$$\text{(i.) } e^{1+x^2}, \quad \text{(ii.) } x^2e^{ax^2}, \quad \text{(iii.) } x^m e^{ax^n}, \quad \text{(iv.) } x^m a^{x^n}.$$

3. Find the differential coefficients of

$$\text{(i.) } x^m\log x, \quad \text{(ii.) } \log\left(\frac{x+1}{x+2}\right), \quad \text{(iii.) } \log(\sqrt{x+1}+\sqrt{x-1}, \quad \text{(iv.) } \log\left(\frac{1-x^2}{4-x^2}\right),$$

$$\text{(v.) } \log\left(\frac{x}{\sqrt{x^2+1}-x}\right), \quad \text{(vi.) } \log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right).$$

4. Differentiate the following expressions logarithmically :—

$$\text{(i.) } \sqrt{(2x+1)(x-2)}, \quad \text{(ii.) } \frac{x}{\sqrt{a^2 \pm x^2}}, \quad \text{(iii.) } \frac{1}{x^2(x-1)^3}, \quad \text{(iv.) } x^x,$$

$$\text{(v.) } \frac{\sin^n mx}{\cos^m nx}, \quad \text{(vi.) } \left(1+\frac{1}{x}\right)^x;$$

and point out why we cannot apply our formula for the differential coefficient of x^n to the case of x^x.

5. If $y=\frac{1}{\sqrt{ac-b^2}}\tan^{-1}\left(\frac{ax+b}{\sqrt{ac-b^2}}\right)$, prove that $\frac{dy}{dx}=\frac{1}{ax^2+2bx+c}$. $(ac>b^2)$

6. If $y=\frac{1}{6}\log\frac{(x+1)^3}{(x^3+1)}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$, prove that $\frac{dy}{dx}=\frac{1}{x^3+1}$.

7. If $y=2\cos^{-1}\sqrt{\frac{\alpha-x}{\alpha-\beta}}$, prove that $\frac{dy}{dx}=\frac{1}{\sqrt{(\alpha-x)(x-\beta)}}$. $(\alpha>x>\beta)$.

8. If $y=\frac{2}{\sqrt{\beta-\alpha}}\cos^{-1}\sqrt{\frac{\alpha-x}{\beta-x}}$, prove that $\frac{dy}{dx}=\frac{1}{(\beta-x)\sqrt{\alpha-x}}$.
$(x<\alpha<\beta)$

9. If $y=\log\left(\frac{b+a\cos x+\sqrt{b^2-a^2}\sin x}{a+b\cos x}\right)$, prove that $\frac{dy}{dx}=\frac{\sqrt{b^2-a^2}}{a+b\cos x}$.
$(b^2>a^2)$

10. If $r=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right\}$, prove that $\frac{dr}{d\theta}=\frac{1}{a+b\cos\theta}$.
$(a^2>b^2)$

11. In the curves whose equations in polar co-ordinates are
(i.) $r=ae^{\theta\cot\alpha}$, (ii.) $r^n=a^n\sin n\theta$, (iii.) $r^n=a^n\cos n\theta$,
(iv.) $r^n=a^n\sec n\theta$, (v.) $r^n=a^n\operatorname{cosec} n\theta$,
find $r\frac{d\theta}{dr}$. Can you give any geometrical meaning to this expression?

12. If $y=e^{-2x}\sin(2x+1)$, prove that $\frac{dy}{dx}=2\sqrt{2}\,.\,e^{-2x}\cos\left(2x+1+\frac{\pi}{4}\right)$.

13. Find the value of $\frac{dy}{dx}$ in the following curves; discuss the way in which it changes as x passes along the axis; and find the turning-points, if there are any, of each curve:—

(i.) $y=x(x-1)^2$.

(ii.) $y=x^2(x-1)^3$.

(iii.) $y=(x-1)^2(x-2)^2$.

(iv.) $y=\frac{x^2+x+1}{x}$.

(v.) $y=\frac{x^2-x+1}{x^2+x+1}$.

(vi) $y=\frac{(x-1)(x-2)}{x^2+x+1}$.

(vii.) $y=\frac{x^2+x+1}{(x-1)(x-2)}$.

(viii.) $y=\frac{(x-1)(x-3)}{(x-2)(x-4)}$.

(ix.) $y=\frac{(x-1)(x-2)}{(x-4)}$.

(x.) $y=\frac{x^3+1}{x^2}$.

[These curves are discussed algebraically and drawn to scale in Chrystal's *Introduction to Algebra*, pp. 391-404. The student is recommended to compare his results with those to be deduced from these figures.]

14. If $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$.

15. If $z=\tan^{-1}\left(\frac{2xy}{x^2+y^2}\right)$, prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$.

16. The formula for the index of refraction μ of a gas at temperature $\theta°$ and pressure p is

$$\mu-1=\frac{\mu_0-1}{1+\alpha\theta}\cdot\frac{p}{760},$$

where μ_0 = the index of refraction at 0°,
α = the coefficient of expansion of the gas.

Prove that the effect of small variations $\delta\theta$ and δp of the temperature and pressure on the index of refraction is to cause it to vary by an amount

$$\delta\mu=\frac{\mu_0-1}{760}\left(\frac{\delta p}{1+\alpha\theta}-\frac{p\alpha\delta\theta}{(1+\alpha\theta)^2}\right).$$

17. If $pv=R\theta$ is the ordinary gas equation, where $\theta=1+\alpha t$, write down the values of

(i.) $\frac{\partial p}{\partial v}$,

(ii.) $\frac{\partial p}{\partial \theta}$,

(iii.) The approximate increase in the pressure due to a small decrease in the volume, the temperature being unchanged,

(iv.) The approximate increase in the volume due to a small increase in the temperature, the pressure remaining the same,

(v.) The approximate increase in the pressure due to a small increase in both temperature and volume.

18. Assuming that the H.P. required to propel a steamer of a given design varies as the square of the length and the cube of the speed, prove that a 2% increase in length, with a 7% increase in H.P., will result in a 1% increase in speed.

19. The area of a triangle is calculated from measurements of two sides and their included angle. Determine the error in the area arising from small errors in these measurements.

20. Assuming that the area of an ellipse whose semiaxes are a and b inches is πab sq. in., and that an elliptical metal plate is expanding by heat or pressure, so that when the semiaxes are 4 and 6 inches, each is increasing at the rate ·1 in. per second, prove that the area of the plate is increasing at the rate of π sq. in. per second.

CHAPTER VI

THE CONIC SECTIONS*

§ 43. **Introductory.**

In this chapter we shall very briefly examine the properties of the Conic Sections, or the curves in which a plane cuts a Right Circular Cone. It is shown in the Geometry of Conics that these curves are the loci of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line. The fixed point S is called the focus; the fixed line, the directrix; and the constant ratio, e, the eccentricity.

When $e<1$, the curve is called an Ellipse;
when $e=1$, the curve is called a Parabola;
when $e>1$, the curve is called a Hyperbola;
and the circle is a special case of the ellipse, the eccentricity being zero, and the directrix at infinity.

§ 44. **The Parabola** ($e=1$).

(i.) *To find its equation.*

Let the focus S be the point $(a, 0)$, and the directrix the line $x+a=0$ (Fig. 19).

Let P be the point (x, y).

Then since $SP^2=PM^2$,

$$(x-a)^3+y^2=(x+a)^2.$$

$$\therefore\ y^2=4ax.$$

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* The student is referred for a fuller discussion of the properties of the Conic Sections to the books mentioned on p. 15. Many of their properties are most easily obtained geometrically, and are to be found in books on *Geometrical Conics.*

This is the equation of the parabola with the origin at the point where the curve cuts the perpendicular from S on the directrix. This point is called the vertex of the curve; the axis

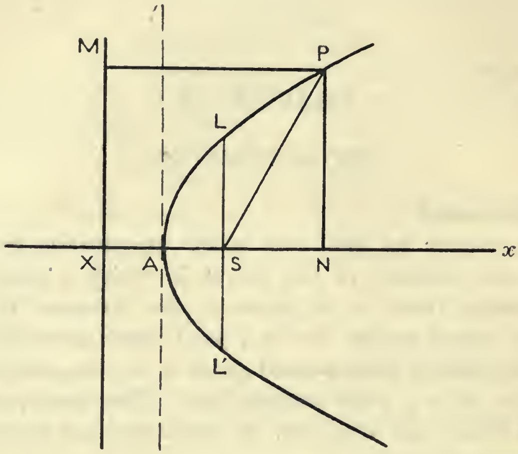

FIG. 19.

of x is called the axis of the curve; and the ordinate L′SL through the focus is called the Latus Rectum.

(ii.) *The shape of the curve.*

From the form of the equation of the curve we see that the curve lies wholly to the right of the axis of y, and that it is symmetrical with regard to the axis of x.

Also since
$$y\frac{dy}{dx}=2a,$$

$$\frac{dy}{dx}=\frac{2a}{y}=\sqrt{\frac{a}{x}}, \text{ when } y>0.$$

It follows that the tangent at the vertex coincides with the axis of y, and that as we move along this branch of the curve in the direction of x increasing, the curve continually ascends, the slope getting less and less the greater x becomes.

(iii.) *The equations of the tangent and normal at* (x_0, y_0).

Since the value of $\frac{dy}{dx}$ at (x_0, y_0) is $\frac{2a}{y_0}$, the equation of the tangent there is

$$\frac{y-y_0}{x-x_0}=\frac{2a}{y_0},$$

or

$$y_0(y-y_0)=2a(x-x_0).$$

This becomes $yy_0=2a(x+x_0)$, since $y_0^2=4ax_0$.

Also the normal is the line

$$y_0(x-x_0)+2a(y-y_0)=0,$$

since this line passes through (x_0, y_0) and is perpendicular to the tangent.

EXAMPLES ON THE PARABOLA

1. Show that the curves $x^2=\pm 4y$ are parabolas, and plot the curves.

2. Show that the equation $y=ax^2+2bx+c$ always represents a parabola, and plot the curves

(i.) $y=x^2+4x+3$,

(ii.) $4y=x^2+4x-8$,

(iii.) $x=y^2+y$.

Find also (i.) The co-ordinates of their foci;

(ii.) The co-ordinates of their vertices;

(iii.) The equations of their latera recta;

(iv.) The lengths of their latera recta;

(v.) The equations of their axes;

(vi.) The equations of the tangents at their vertices.

3. Find algebraically and graphically the minimum value of the expression x^2-2x-4, and the maximum value of $5+4x-2x^2$.

4. The tangent at P meets the axis of the parabola of Fig. 19 in T, and the normal meets the axis in G. Prove the following properties:—

(i.) AN = AT,

(ii.) SP = ST = SG,

(iii.) NG = 2AS,

and show that the tangents at the ends of a focal chord meet at right angles on the directrix.

5. Prove that the line $y=x+1$ touches the parabola $y^2=4x$, and that the line $y=mx+\frac{a}{m}$ touches the parabola $y^2=4ax$. Find the point of contact in each case.

6. Find the equations of the tangent and normal at the point where the line $x=2$ cuts the parabola $x^2=4y$.

7. Find the equations of the tangents and normals at the extremities of the latus rectum of the parabola $y^2=4ax$, and show that they form a square.

8. Prove that the locus of the middle points of the chords of the parabola $y^2=4ax$, which make an angle θ with the axis of x, is the straight line

$$y=2a\cot\theta.$$

9. The chord PQ meets the axis of the parabola of Fig. 19 in O. PM and QN are the ordinates of P and Q. Prove that $AM.AN=AO^2$, by finding the equation of the chord in its simplest form.

10. The position of a moving point is given by the equations

$$x=v\cos\alpha.t,$$
$$y=v\sin\alpha.t-\tfrac{1}{2}gt^2.$$

Interpret the equations, and prove that the point moves on a parabola

whose axis is parallel to the axis of y;

whose vertex is at the point $\left(\frac{v^2\sin\alpha\cos\alpha}{g}, \frac{v^2\sin^2\alpha}{2g}\right)$;

whose directrix is the line $y=\frac{v^2}{2g}$;

and whose latus rectum is of length $\frac{2v^2\cos^2\alpha}{g}$.

§45. **The Ellipse** ($e<1$).

(i.) *To find its equation.*

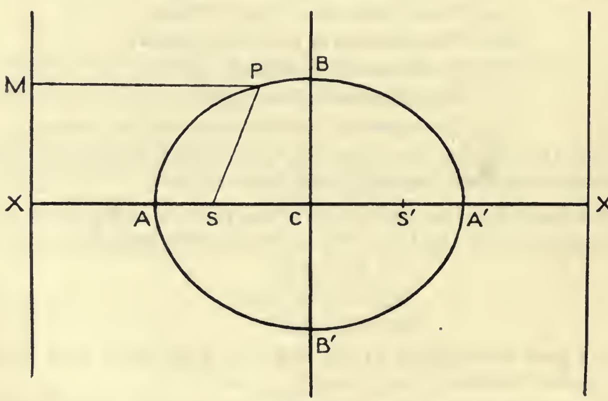

Fig. 20.

Let the axis of x be the axis of the ellipse (*i.e.* the line through the focus perpendicular to the directrix).

Let S be the point $(d, 0)$;

and let the axis of y be the directrix.

Let $P(x, y)$ be any point upon the curve.

Then $$SP^2 = e^2\, PM^2.$$

$$\therefore\ (x-d)^2 + y^2 = e^2x^2.$$

$$\therefore\ x^2(1-e^2) - 2xd + y^2 = -d^2.$$

$$\therefore\ \left(x - \frac{d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{d^2}{(1-e^2)^2} - \frac{d^2}{1-e^2} = \frac{d^2e^2}{(1-e^2)^2}.$$

Now change the origin to the point $\left(\frac{d}{1-e^2}, 0\right)$, keeping the axes parallel to their original directions.

The equation of the ellipse then becomes

$$x^2 + \frac{y^2}{(1-e^2)} = \frac{d^2e^2}{(1-e^2)^2};$$

i.e. $$\frac{x^2}{\dfrac{d^2e^2}{(1-e^2)^2}} + \frac{y^2}{\dfrac{d^2e^2}{(1-e^2)}} = 1.$$

Putting $$a^2 = \frac{d^2e^2}{(1-e^2)^2}$$

and $$b^2 = \frac{d^2e^2}{1-e^2},$$

we have $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad b^2 = a^2(1-e^2).$$

In this form the origin C is called the centre of the curve, since it bisects every chord which passes through it. This is clear, since if (x_1, y_1) lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so does $(-x_1, -y_1)$.

Also we notice that $\text{CS} = \frac{d}{1-e^2} - d = \frac{de^2}{1-e^2} = ae,$

and that $$\text{CX} = \frac{d}{1-e^2} = \frac{a}{e}.$$

From the symmetry of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

it is clear that there is another focus, namely, the point $(ae, 0)$; and another directrix, the line $x = \frac{a}{e}$, with regard to the axes through the point C.

The axis of x is in this case called the major axis, and the axis of y the minor axis. The one is of length $2a$; the other of length $2b$. If b had been greater than a, the foci would have lain upon the axis of y, and this axis would have been the major axis. When a and b are known, the eccentricity e is given by

$$b^2 = a^2(1 - e^2). \quad (a > b)$$

In the circle $a = b$, and $e = 0$.

(ii.) *The shape of the curve.*

Since the equation involves only the terms x^2 and y^2, the curve is symmetrical about both the axes of x and y.

Also, since $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$, we see that x must lie between $-a$ and $+a$, and that, as x passes from $-a$ to $+a$, the positive value of y gradually increases from zero to b, and then diminishes again to zero.

The curve is thus a closed curve, lying altogether within the rectangle $x = \pm a$, $y = \pm b$.

This is also evident from the property of Ex. 3, p. 87, where it is proved that the curve may be drawn by fixing the two ends of a string of length $2a$ to the points S and S′, and holding the string tight by the point P of the tracing pencil.

(iii.) *The equations of the tangent and normal at* (x_0, y_0).

Since
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0.$$

Therefore the equation of the tangent at (x_0, y_0) is

$$\frac{y - y_0}{x - x_0} = -\frac{b^2 x_0}{a^2 y_0},$$

which becomes
$$(x - x_0)\frac{x_0}{a^2} + (y - y_0)\frac{y_0}{b^2} = 0;$$

or
$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1, \quad \text{since } \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

It follows that the equation of the normal is

$$(x-x_0)\frac{y_0}{b^2}-(y-y_0)\frac{x_0}{a^2}=0,$$

or

$$\frac{xy_0}{b^2}-\frac{yx_0}{a^2}=x_0y_0\left(\frac{a^2-b^2}{a^2b^2}\right),$$

or

$$\frac{a^2x}{x_0}-\frac{b^2y}{y_0}=a^2-b^2.$$

EXAMPLES ON THE ELLIPSE

1. Trace the ellipses (i.) $3x^2+4y^2=12$;
(ii.) $3(x-1)^2+4(y-2)^2=12$;
(iii.) $x^2+4y^2=8y$;
(iv.) $4x^2+3y^2=12$;

and find the co-ordinates of the foci and of the extremities of the axes, the length of the latus rectum, and the eccentricity of each.

2. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that the co-ordinates of any point may be expressed as $x=a\cos\theta$, $y=b\sin\theta$; and interpret the result geometrically.

3. P is the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that $SP=a+ex_1$ and $S'P=a-ex_1$, and deduce that the curve is the locus of a point which moves so that the sum of its distances from two fixed points is constant.

4. The tangent at P meets the major axis in T, and PN is the ordinate of P; prove that $CN \cdot CT=CA^2$.

5. The normal at P meets the major axis in G. Prove that $SG:SP=e$, and deduce that PG bisects the angle SPS′.

6. Prove that the middle point of the chord $y=x+1$ lies upon $y=-\frac{b^2}{a^2}x$, and that the middle points of chords parallel to $y=mx$ lie upon the chord $y=m'x$, where $mm'+\frac{b^2}{a^2}=0$.

7. If CP bisects chords parallel to CD, prove that CD bisects chords parallel to CP (CP and CD are then said to be *conjugate diameters*); and prove that the tangents at P and D form with CP and CD a parallelogram.

8. If P is the point $(a\cos\theta, b\sin\theta)$, prove that CD is the line

$$a\sin\theta y+b\cos\theta x=0,$$

and deduce that $CP^2+CD^2=a^2+b^2$.

§ 46. **The Hyperbola** ($e > 1$).

(i.) *To find its equation.*

Proceeding as in § 45 (i.), we obtain the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where we have written a^2 for $\dfrac{d^2e^2}{(e^2-1)^2}$

and b^2 for $\dfrac{d^2e^2}{e^2-1}$, *i.e.* for $a^2(e^2-1)$.

Also d is the distance from the focus S to the directrix.

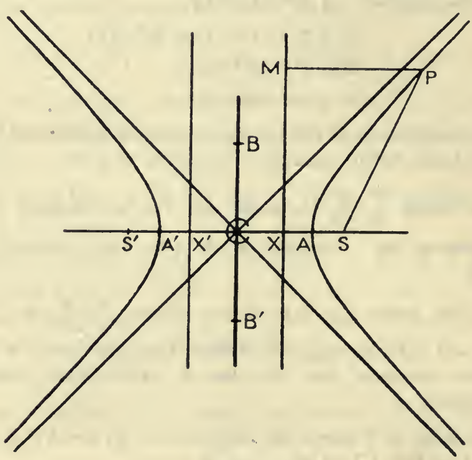

FIG. 21.

It follows that $CS = ae$, $CX = \dfrac{a}{e}$, and that there are two foci and two directrices.

The line joining the foci S, S′ is called the transverse axis of the hyperbola.

(ii.) *The shape of the curve.*

The form of the equation shows that the curve is symmetrical about both axes. Also since $y^2 = b^2\left(\dfrac{x^2}{a^2} - 1\right)$, it is clear that x cannot lie between $-a$ and $+a$; since $x^2 = a^2\left(1 + \dfrac{y^2}{b^2}\right)$, y can have any value whatsoever.

If we write the equation as

$$\frac{y^2}{x^2}=\frac{b^2}{a^2}-\frac{b^2}{x^2},$$

we see that, when x is numerically very great, $\frac{y^2}{x^2}$ is less than, but very nearly equal to $\frac{b^2}{a^2}$; and that for all points on the curve $\frac{y^2}{x^2}$ is less than $\frac{b^2}{a^2}$.

Also the positive value of y decreases as x passes from $-\infty$ to $-a$, where it vanishes; and it increases without limit from the value zero at $x=a$, as x passes along the positive axis of x.

The shape of the curve is thus as in Fig. 21. The lines $y=\pm\frac{b}{a}x$ are called the asymptotes, and the curve lies wholly between those lines; while, as the numerical value of x gets greater and greater, it approaches more and more nearly to these lines, without ever actually reaching them.

(iii.) *The equations of the tangent and normal at* (x_0, y_0) are easily shown to be

$$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1,$$

and

$$\frac{y_0}{b^2}(x-x_0)+\frac{x_0}{a^2}(y-y_0)=0.$$

(iv.) *The product of the perpendiculars from any point on the curve to the asymptotes is constant.*

The asymptotes are the lines $y=\pm\frac{b}{a}x$. Then if PM, PN are the perpendiculars to these lines from the point (x_0, y_0),

$$\text{PM}=\frac{-y_0+\frac{b}{a}x_0}{\sqrt{1+\frac{b^2}{a^2}}},\quad \text{PN}=\frac{y_0+\frac{b}{a}x_0}{\sqrt{1+\frac{b^2}{a^2}}}.$$

Therefore $$\text{PM}\,.\,\text{PN}=\frac{b^2x_0^2-a^2y_0^2}{a^2+b^2}=\frac{a^2b^2}{a^2+b^2},$$

since $$\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1.$$

Hence PM . PN = constant.

When $b^2 = a^2$, the asymptotes are at right angles, and the eccentricity is $\sqrt{2}$. In this case, by taking the asymptotes as axes, the equation $x^2 - y^2 = a^2$ is transformed to

$$2xy = a^2.$$

This equation is of the form $xy = c^2$, a relation which is of the greatest importance in Physics. We could obtain an equation of the same form for any hyperbola referred to its asymptotes as oblique axes.

EXAMPLES ON THE HYPERBOLA

1. Trace the hyperbolas:

$$\text{(i.) } 3x^2 - 4y^2 = 12,$$
$$\text{(ii.) } 3(x-1)^2 - 4(y-2)^2 = 12,$$
$$\text{(iii.) } x^2 - 4y^2 = 8y,$$
$$\text{(iv.) } 4x^2 - 3y^2 = 12;$$

and find the co-ordinates of the foci and of the points where each curve cuts its transverse axis, the length of the latus rectum, and the eccentricity of each.

2. Trace the rectangular hyperbolas:

$$\text{(i.) } xy = \pm 4,$$
$$\text{(ii.) } y = 1 \pm \frac{1}{x},$$

and find the co-ordinates of the foci and of the points where the transverse axis meets each curve.

3. Prove that the tangent at (x_0, y_0) to the hyperbola $xy = c^2$ is $xy_0 + yx_0 = 2c^2$, and that the point of contact bisects the part of the tangent cut off by the asymptotes.

4. P is the point (x_1, y_1) on the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Prove that $SP = ex_1 - a$, and $S'P = ex_1 + a$, and deduce that the curve is the locus of a point which moves so that the difference of its distances from two fixed points is constant.

5. The tangent at P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis in T, and PN is the ordinate of P. Prove that $CN \, . \, CT = a^2$.

6. The normal at P meets the major axis in G; show that $SG = eSP$, and deduce that PG bisects the angle SPS′.

7. Prove that, in the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the middle point of the chord $y=x+1$ lies upon the line $y=\frac{b^2}{a^2}x$, and that the locus of the middle points of chords parallel to $y=mx$ is the line $y=m'x$, where $mm'=\frac{b^2}{a^2}$.

8. If CP and CD are two conjugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (*i.e.* if each bisects chords parallel to the other), prove that if P lies upon this curve, CD does not meet the curve, and that if D is the point where CD meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$,

$$CP^2-CD^2=a^2-b^2.$$

CHAPTER VII

THE INTEGRAL CALCULUS—INTEGRATION

§ 47. Introductory.

In considering the motion of a point along a straight line, we saw that, if

$$s = f(t)$$

is the relation between the distance and the time, the velocity v is given by

$$v = \frac{ds}{dt} = f'(t).$$

In general, the problem of the Differential Calculus is as follows: given the law in obedience to which two related magnitudes vary, to find the rate at which the one changes with regard to the other. The problem of the Integral Calculus is the inverse one: given the rate at which the magnitudes change with regard to each other, to find the law connecting them. In other words, in the Differential Calculus we determine the infinitesimal change in the one magnitude which corresponds to an infinitesimal change in the other, when we know what function the one is of the other. In the Integral Calculus we determine what function the one is of the other, when the corresponding infinitesimal changes are known. We have thus to find the function of x, denoted by y, which is such that

$$\frac{dy}{dx} = f(x).$$

The value of y which satisfies this equation is written $\int f(x)\,dx$, and is called *the integral of* $f(x)$ *with regard to* x. When we have found the *integral of* $f(x)$, we are said to have integrated the function. The process of finding the integral is called integration.

Ex. 1. $\int \frac{dx}{x+a} = \log(x+a)$, since $\frac{d}{dx}\log(x+a) = \frac{1}{x+a}$.

2. $\int e^{ax}dx = \frac{e^{ax}}{a}$, since $\frac{d}{dx}\frac{e^{ax}}{a} = e^{ax}$.

3. Fill up the blanks in the following table :—

$f(x)$	1	x	x^2	x^3	x^4	x^5	x^{10}	x^n
$\int f(x)\,dx$								

n being a positive integer.

4. Fill up the blanks in the following table :—

$f(x)$	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{1}{x^3}$	$\frac{1}{x^4}$	$\frac{1}{x^5}$	$\frac{1}{x^{10}}$	$\frac{1}{x^{n+1}}$
$\int f(x)\,dx$							

$n+1$ being a positive integer.

5. Verify the following results :—

$f(x)$	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
$\int f(x)\,dx$	$-\cos x$	$\sin x$	$\log\sec x$	$\log\sin x$	$\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$	$\log\tan\frac{x}{2}$

6. Verify the following results :—

$f(x)$	$\sin^2 x$	$\cos^2 x$	$\tan^2 x$	$\cot^2 x$	$\sec^2 x$	$\operatorname{cosec}^2 x$
$\int f(x)\,dx$	$\frac{x}{2}-\frac{\sin 2x}{4}$	$\frac{x}{2}+\frac{\sin 2x}{4}$	$\tan x - x$	$-\cot x - x$	$\tan x$	$-\cot x$

In each of these cases we might have added any constant to the answer, since the differential coefficient of a constant is zero, and the complete result in the first two examples would have been

$$\int \frac{dx}{x+a} = \log(x+a) + C,$$

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C,$$

where C is called the constant of integration.

It is thus evident that the equations

$$\frac{d}{dx}\mathrm{F}(x)=f(x)$$

and

$$\mathrm{F}(x)=\int f(x)\,dx$$

represent the same thing, and that the fuller statement of the second would be

$$\mathrm{F}(x)+\mathrm{C}=\int f(x)\,dx.$$

Owing to the presence of the arbitrary constant, $\int f(x)\,dx$ is called the Indefinite Integral of $f(x)$.

The geometrical meaning of the constant of integration is that there is a family of curves all having the same slope as a given curve. The curves

$$y=\mathrm{F}(x)+\mathrm{C}$$

are all parallel, when C is given different constant values.

§ 48. Table of Standard Integrals.

When integration is regarded in this way,* the first thing we have to do is to draw up a list of the most important integrals. This table is obtained from the corresponding results in differentiation. Any result in integration can always be verified by differentiation. Later we shall see that there are certain general theorems on integration which correspond to the general theorems of differentiation. These will help us to decide upon the most likely ways of finding an answer to the question which the symbol of integration puts to us; namely, *What is the function whose differential coefficient is the given expression?* To answer this question is in very many cases impossible; but practice soon makes it easy to recognise the simple cases which can be treated with success.

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* In Chapter VIII. we shall learn that there is another way of looking at integration: that, in fact, integration is simply a summation, or more exactly an integral is the limit of a sum: and that the symbol $\int$ of integration stands for a capital S, denoting that a sum is being taken.

The following is the table of Standard Forms :—

(i.) $\int x^n dx = \frac{x^{n+1}}{n+1}$, since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$, $(n \neq -1)$

(ii.) $\int \frac{dx}{x} = \log x$, since $\frac{d}{dx}(\log x) = \frac{1}{x}$, $(x > 0)$

(iii.) $\int e^{ax} dx = \frac{1}{a} e^{ax}$,

(iv.) $\int a^x dx = \frac{1}{\log a} a^x$,

(v.) $\int \cos x dx = \sin x$,

(vi.) $\int \sin x dx = -\cos x$,

(vii.) $\int \tan x dx = \log(\sec x)$,

(viii.) $\int \operatorname{cosec} x dx = \log\left(\tan \frac{x}{2}\right)$,

(ix.) $\int \sec^2 x dx = \tan x$,

(x.) $\int \operatorname{cosec}^2 x dx = -\cot x$,

(xi.) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$ or $\left(-\cos^{-1} \frac{x}{a}\right)$, $(a^2 > x^2)$

(xii.) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$ or $-\frac{1}{a} \cot^{-1} \frac{x}{a}$,

(xiii.) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$, $(x^2 > a^2)$

(xiv.) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$, $(x^2 < a^2)$

(xv.) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$.

(The logarithms are to the base e, and the angles are measured in radians.)

The student is recommended to draw up a corresponding table for the cases where $mx + n$ takes the place of x in this list.

§ 49. **Two General Theorems.**

$$\text{(i.)} \quad \int (cu)\,dx = c\int u\,dx,$$

$$\text{(ii.)} \quad \int (u+v)\,dx = \int u\,dx + \int v\,dx,$$

c being a constant, and u, v functions of x.

In the first theorem, the left-hand side of the equation asks us a question: What is the function whose differential coefficient is cu?

The right-hand side tells us that the answer to this question is $c\int u\,dx$.

We need only verify this answer.

To do so we differentiate $c\int u\,dx$.

We have $$\frac{d}{dx}c\int u\,dx = c\frac{d}{dx}\int u\,dx = cu,$$

since differentiating an integral simply cancels the symbol of integration and the dx.

It follows that $$\int cu\,dx = c\int u\,dx.$$

In the second theorem, the left-hand side of the equation asks us a question: What is the function whose differential coefficient is $u+v$?

The right-hand side tells us that the answer to this question is $\int u\,dx + \int v\,dx$.

We need only verify this answer.

To do so we differentiate $\int u\,dx + \int v\,dx$.

We have $$\frac{d}{dx}\left(\int u\,dx + \int v\,dx\right) = \frac{d}{dx}\int u\,dx + \frac{d}{dx}\int v\,dx$$
$$= u + v.$$

It follows that $$\int (u+v)\,dx = \int u\,dx + \int v\,dx.$$

Using these theorems we can readily integrate any ordinary algebraical expression of the form

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a^n.$$

Also, if we call this expression $f(x)$, we can integrate

$$\frac{f(x)}{x-a}.$$

We have only to divide $f(x)$ by $(x-a)$, and integrate each term of the quotient. If there is a remainder, the corresponding term in the answer involves $\int\frac{dx}{x-a}$, or $\log(x-a)$.

If there are several factors, the work in Algebra on Partial Fractions comes to our aid. If the numerator is of the same or of higher degree than the denominator, it must be divided by the denominator before finding the partial fractions.

Ex. 1. $$\int(ax^2+2bx+c)\,dx = a\int x^2dx + 2b\int x\,dx + c\int 1\,dx^*$$
$$= \frac{ax^3}{3} + bx^2 + cx.$$

2. $$\int\frac{2x+1}{2x-1}dx = \int\left(1+\frac{2}{2x-1}\right)dx$$
$$= \int dx + 2\int\frac{dx}{2x-1}$$
$$= x + \log(2x-1).$$

3. $$\int\frac{dx}{x^2-a^2} = \int\frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx\,\dagger$$
$$= \frac{1}{2a}\int\frac{dx}{x-a} - \frac{1}{2a}\int\frac{dx}{x+a}$$
$$= \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right), \quad \text{when} \quad x > a.$$

4. $$\int \sin^2 x\,dx = \int\frac{1-\cos 2x}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int\cos 2x\,dx$$
$$= \frac{x}{2} - \frac{\sin 2x}{4}.$$

5. $$\int\cos^2 x\,dx = \int\frac{1+\cos 2x}{2}dx = \frac{x}{2} + \frac{\sin x\cos x}{2}.$$

---

* $\int 1\,dx$ is usually written as $\int dx$.

† This is one of the standard forms.

6. Fill up the blanks in the following table :—

$f(x)$	$2x+\frac{1}{x}$	$\frac{x^2+x+1}{x^2}$	$3x^2+2x+\frac{1}{x+1}$	$3x^2+\frac{1}{x^2+1}$	$3x^2+\frac{1}{x^2-1}$	$\frac{x+1}{x+2}$	$\frac{x^2+2x+1}{x+2}$
$\int f(x)\,dx$							

Many of the most important applications of the Integral Calculus involve only such integrals as we have now learned to calculate. The student, who has not time to take up the question of integration more fully, could omit, in the meantime, the remaining articles of this chapter.

§ 50. **Integration by Substitution.**

To prove that $\int f(x)\,dx = \int f(x)\frac{dx}{dt}dt$, *where* $x=\phi(t)$.

This important result, which allows us to change an integral with regard to x into an integral in terms of another variable, may be deduced at once from the rule for differentiating a function of a function.

Let $$y=\int f(x)\,dx \quad \text{and} \quad x=\phi(t).$$

From the relation between x and t, y is a function of t.

But $$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}.$$

$$\therefore\ \frac{dy}{dt}=f(x)\frac{dx}{dt}, \quad \text{since} \quad \frac{dy}{dx}=f(x).$$

$$\therefore\ y=\int f(x)\frac{dx}{dt}\,dt,$$

by the definition of an integral.

The expressions under the sign of integration are supposed given in terms of t.

This result may be written

$$\text{(A)} \qquad \int f(x)\,dx=\int f(x)\frac{dx}{dt}\,dt=\int f[\phi(t)]\frac{d}{dt}[\phi(t)]\,dt.$$

The simple rule for "changing the variable" from x to t is:

Replace dx *by* $\frac{dx}{dt}dt$, *and by means of the equation connecting* x *and* t, *express* $f(x)$ *as a function of* t.

The advantages of this method will be evident from the following examples:—

Ex. 1. $\int (ax+b)^n dx.$ Put $ax+b=u.$

$$\therefore \frac{dx}{du}=\frac{1}{a}.$$

Thus $\int (ax+b)^n dx=\int u^n \frac{1}{a}\,du=\frac{1}{a}\int u^n du=\frac{u^{n+1}}{a(n+1)}=\frac{1}{a(n+1)}(ax+b)^{n+1}.$

Similarly 2. $\int \sin(ax+b)\,dx=\frac{1}{a}\int \sin u\,du=-\frac{\cos u}{a}=-\frac{1}{a}\cos(ax+b).$

3. $\int \frac{dx}{\sqrt{a^2x^2-b^2}}.$ Put $ax=u.$

$$\therefore \frac{dx}{du}=\frac{1}{a}.$$

Thus
$$\int \frac{dx}{\sqrt{a^2x^2-b^2}}=\int \frac{1}{\sqrt{u^2-b^2}}\frac{1}{a}\,du=\frac{1}{a}\int \frac{du}{\sqrt{u^2-b^2}}$$
$$=\frac{1}{a}\log(u+\sqrt{u^2-b^2})$$
$$=\frac{1}{a}\log(ax+\sqrt{a^2x^2-b^2}).$$

4. $\int \frac{\log x}{x}\,dx.$ Put $x=e^u.$

$$\therefore \frac{dx}{du}=e^u.$$

Thus
$$\int \frac{\log x}{x}\,dx=\int \frac{u}{e^u}e^u\,du=\int u\,du=\frac{1}{2}u^2$$
$$=\frac{1}{2}(\log x)^2.$$

5. $\int \frac{dx}{(1-x)\sqrt{1-x^2}}.$ Put $x=\cos\theta.$

$$\therefore \frac{dx}{d\theta}=-\sin\theta.$$

Thus
$$\int \frac{dx}{(1-x)\sqrt{1-x^2}}=\int \frac{1}{(1-\cos\theta)\sin\theta}(-\sin\theta)\,d\theta$$
$$=-\frac{1}{2}\int \frac{d\theta}{\sin^2\frac{\theta}{2}}$$
$$=\cot\frac{\theta}{2}$$
$$=\frac{\sqrt{1-x^2}}{1-x}.$$

6. Integrate the following expressions :—

(α) $x^{n-1}(ax^n+b)$; $x\sqrt{a^2+x^2}$; $\dfrac{x^2}{1+x^6}$.

(β) $\dfrac{1}{x^2+2x+2}$; $\dfrac{x+1}{x^2+2x+2}$, putting $x+1=u$.

(γ) $\dfrac{1}{ax^2+2bx+c}$; $\dfrac{ax+b}{ax^2+2bx+c}$, putting $ax+b=u$. $(ac>b^2)$

(δ) $\dfrac{1}{\sqrt{x^2+4x+5}}$; $\dfrac{x+2}{\sqrt{x^2+4x+5}}$, putting $x+2=u$.

(ε) $\dfrac{1}{\sqrt{ax^2+2bx+c}}$; $\dfrac{ax+b}{\sqrt{ax^2+2bx+c}}$, putting $ax+b=u$. $(ac>b^2)$

(ζ) $\sin^3 x\cos^3 x$; $\dfrac{\cos x}{a+b\sin x}$; $\cot x$, putting $\sin x=u$.

(η) $\dfrac{1}{a^2\cos^2 x+b^2\sin^2 x}$; $\dfrac{1}{\cos^2 x\sin^2 x}$, putting $\tan x=u$.

§ 51. **Integration by Substitution**—*continued.*

Although there are certain general principles that guide us in the choice of a suitable substitution, a second form (B) of the theorem of § 50 will often suggest what the transformation should be. We have seen that

$$\int f[\phi(t)]\frac{d}{dt}[\phi(t)]dt=\int f(x)\,dx, \text{ where } x=\phi(t).$$

We may write this result in the form

$$\text{(B)}\qquad \int f[\phi(x)]\frac{d}{dx}[\phi(x)]\,dx=\int f(u)\,du, \text{ where } u=\phi(x),^*$$

as the particular symbol we employ is immaterial.

Thus in the case of the examples of last article we obtain our results immediately—

e.g. (i.) $\displaystyle\int(ax+b)^n dx=\frac{1}{a}\int(ax+b)^n\frac{d}{dx}(ax+b)\,dx$

$$=\frac{1}{a}\int u^n du, \text{ where } ax+b=u,$$

$$=\frac{1}{n+1}\frac{u^{n+1}}{a}=\frac{1}{a(n+1)}(ax+b)^{n+1}.$$

---

* This can be verified by starting with

$$\int f(u)\,du,$$

and putting $u=\phi(x)$, as in (A).

(ii.) $$\int \sin^2 x \cos x\, dx = \int \sin^2 x \frac{d}{dx}(\sin x)\, dx$$

$$= \int u^2 du, \text{ where } \sin x = u,$$

$$= \frac{1}{3}\sin^3 x.$$

(iii.) $$\int \frac{x^4}{1+x^5} \cdot dx = \frac{1}{5}\int \frac{1}{1+x^5}\frac{d}{dx}(1+x^5)\, dx$$

$$= \frac{1}{5}\int \frac{du}{u}, \text{ where } u = 1 + x^5,$$

$$= \frac{1}{5}\log u$$

$$= \frac{1}{5}\log(1+x^5).$$

(iv.) $$\int \frac{\mathbf{f'(x)}}{\mathbf{f(x)}}\, \mathbf{dx} = \mathbf{\log f(x)}.$$

In this way it is easy to see that

$$\int \frac{ax+b}{ax^2+2bx+c}\, dx = \frac{1}{2}\log(ax^2+2bx+c),$$

since the integral may be written as

$$\frac{1}{2}\int \frac{1}{ax^2+2bx+c}\frac{d}{dx}(ax^2+2bx+c)\, dx,$$

i.e. $$\frac{1}{2}\int \frac{du}{u}, \text{ where } u = ax^2+2bx+c.$$

Also $$\int \frac{dx}{ax^2+2bx+c} = \int \frac{1}{(ax+b)^2+ac-b^2}\frac{d}{dx}(ax+b)\, dx$$

$$= \int \frac{1}{u^2+ac-b^2}du, \text{ where } u = ax+b,$$

and this is one of the standard forms.

It follows that any expression of the form

$$\frac{lx+m}{ax^2+2bx+c}$$

can be easily integrated, since we can rewrite the numerator as

$$P(ax+b)+Q,$$

where
$$P=\frac{l}{a};\ Q=\frac{am-lb}{a}.$$

If higher powers of x occur in the numerator, we must first of all divide out by the denominator till we obtain a remainder of the first degree or a constant.*

The expression $\dfrac{lx+m}{\sqrt{ax^2+2bx+c}}$ *may be reduced in a similar way.*

Ex. Integrate the following expressions :—

(i.) $\dfrac{1}{x^2\pm 4}$; $\dfrac{1}{a^2x^2\pm b^2}$; $\dfrac{1}{4x^2+4x\pm 3}$; $\dfrac{x+1}{4x^2+4x\pm 3}$; $\dfrac{2x+3}{3+4x-x^2}$;

$\dfrac{x^4}{x^2\pm 1}$; $\dfrac{x^2-x+1}{x^2+x+1}$; $\dfrac{x-1}{x^2-5x+6}$; $\dfrac{x^2+x+1}{(x-1)(x-2)}$.

(ii.) $\dfrac{1}{\sqrt{x^2\pm 4}}$; $\dfrac{1}{\sqrt{a^2x^2\pm b^2}}$; $\dfrac{1}{\sqrt{4x^2+4x\pm 3}}$; $\dfrac{x+1}{\sqrt{4x^2+4x\pm 3}}$; $\dfrac{2x+3}{\sqrt{5+4x-x^2}}$.

§ 52. Integration by Parts.

The second important method in integration is called integration by parts, and can be used only when the function to be integrated is the product of two functions, one of which can be expressed as a differential coefficient. This method follows at once from the rule for the differentiation of a product.

Since
$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx},$$

we have $uv=\displaystyle\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)dx$, by the definition of integration,

$$=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}dx, \text{ by § 49.}$$

It follows that
$$\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx.$$

This result will only be of use if $\displaystyle\int v\frac{du}{dx}dx$ can be more easily evaluated than $\displaystyle\int u\frac{dv}{dx}dx$.

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* When the factors of the denominator are real, the method of Partial Fractions should be employed.

For example—

(i.) $$\int x \log x \, dx = \frac{1}{2}\int \log x \frac{d}{dx}(x^2)\, dx$$

$$= \frac{1}{2}\left(x^2 \log x - \int x^2 \frac{d}{dx}(\log x)\, dx\right)$$

$$= \frac{1}{2}\left(x^2 \log x - \int x \, dx\right)$$

$$= \frac{1}{2}\left(x^2 \log x - \frac{x^2}{2}\right)$$

$$= \frac{x^2}{4}\left(2 \log x - 1\right).$$

(ii.) $$\int x^2 \cos x \, dx = \int x^2 \frac{d}{dx}(\sin x)\, dx$$

$$= x^2 \sin x - \int \sin x \frac{d}{dx}(x^2)\, dx$$

$$= x^2 \sin x - 2\int \sin x \, x \, dx$$

$$= x^2 \sin x + 2\int x \frac{d}{dx}(\cos x)\, dx$$

$$= x^2 \sin x + 2\left(x \cos x - \int \cos x \frac{d}{dx}(x)\, dx\right)$$

$$= x^2 \sin x + 2\left(x \cos x - \int \cos x \, dx\right)$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x.$$

In both of these examples this artifice allows us gradually to reduce the integral to one of a simpler form.

In such cases, where powers of x are associated with a trigonometrical, exponential, or logarithmic term, it is of great value.*

An important expression which can be integrated by this method is $\sqrt{a^2 - x^2}$.

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* Cf. p. 105; Exs. 11, 12, 13, 14, and 15.

We have

$$\int\sqrt{a^2-x^2}\,dx=\int\sqrt{a^2-x^2}\frac{d}{dx}x\,dx$$

$$=x\sqrt{a^2-x^2}-\int x\frac{d}{dx}\sqrt{a^2-x^2}\,dx$$

$$=x\sqrt{a^2-x^2}+\int\frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}.$$

$$\therefore\ 2\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}.$$

$$\therefore\ \int\sqrt{\mathbf{a}^2-\mathbf{x}^2}\,\mathbf{dx}=\frac{\mathbf{x}}{2}\sqrt{\mathbf{a}^2-\mathbf{x}^2}+\frac{\mathbf{a}^2}{2}\sin^{-1}\frac{\mathbf{x}}{\mathbf{a}}.$$

The expressions $\sqrt{x^2-a^2}$ and $\sqrt{a^2+x^2}$ can be integrated in the same way.

Ex. Integrate the following expressions :—

$x^2\log x$; x^3e^x; $x\tan^{-1}x$; $x^2\sin ax$; $\sqrt{9-x^2}$; $\sqrt{x^2-9}$.

EXAMPLES ON CHAPTER VII

1. Integrate the following expressions :—

(i.) $(x-a)^3,\quad \dfrac{1}{\sqrt{ax+b}},\quad \dfrac{1+x}{\sqrt{x}},\quad \dfrac{x+2}{x+3}.$

(ii.) $\dfrac{1}{x(1-x)},\quad \dfrac{2x-1}{x^2-3x+2},\quad \dfrac{x^3}{x^2+x+1},\quad \dfrac{x^4}{x^2-x+1}.$

(iii.) $\dfrac{1}{\sqrt{x(1-x)}},\quad \dfrac{2x-1}{\sqrt{x^2-3x+2}},\quad \dfrac{x+1}{\sqrt{x^2+x+1}}.$

2. Integrate the following expressions by parts :—

$\sin^{-1}x,\quad x^2\tan^{-1}x,\quad x^2\sin 4x,\quad x^2\cos 3x,\quad x^m\log x,\quad x^2e^{-x}.$

3. Prove that $\dfrac{1}{x^3+1}=\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)},$

and hence integrate the expression.

4. Prove that

$$\frac{1}{(x+1)(x-1)^2}=\frac{1}{2(x-1)^2}-\frac{1}{4(x-1)}+\frac{1}{4(x+1)},$$

and hence integrate the expression.

5. Prove that $$\frac{x-1}{(x-2)(x-3)}=\frac{2}{x-3}-\frac{1}{x-2},$$
and hence integrate the expression.

6. Integrate the expressions $x\sqrt{1+x}$ and $\frac{1}{x\sqrt{1+x}}$ by putting $x+1=u^2$.

7. Prove that $$\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{1-x^2}}\quad(\text{put } x=\sin\theta).$$

8. Integrate the following trigonometrical expressions:—
$$\frac{1}{\sin\theta},\quad\frac{1}{\sin(\theta+\alpha)},\quad\frac{1}{\sin\theta+\cos\theta},\quad\frac{1}{\cos^2\theta\sqrt{a^2\tan^2\theta+b^2}},\quad\frac{\sin x}{\cos^2 x(4\tan^2 x+3)}.$$

9. Show that, when $a^2>b^2$,
$$\int\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right).$$
[Put $a+b\cos x$ into the form $(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}$.]

Also integrate the expressions
$$\frac{1}{5\pm4\cos x},\quad\frac{1}{4\pm5\cos x},\quad\frac{1}{3\pm2\sin x},\quad\frac{1}{2\pm3\sin x}.$$

10. Prove, by integration by parts, that
$$\text{(i.)}\ \int e^{ax}\cos bx\,dx=\frac{b\sin bx+a\cos bx}{a^2+b^2}e^{ax},$$
$$\text{(ii.)}\ \int e^{ax}\sin bx\,dx=\frac{a\sin bx-b\cos bx}{a^2+b^2}e^{ax}.$$

11. Prove, by integration by parts, that
$$\int\sin^n\theta\,d\theta=-\frac{\cos\theta\sin^{n-1}\theta}{n}+\frac{n-1}{n}\int\sin^{n-2}\theta\,d\theta,$$
and hence show that
$$\int\sin^4\theta\,d\theta=-\frac{\sin^3\theta\cos\theta}{4}-\frac{3}{8}\sin\theta\cos\theta+\frac{3}{8}\theta.$$

12. Prove, by integration by parts, that
$$\int\cos^n\theta\,d\theta=\frac{\sin\theta\cos^{n-1}\theta}{n}+\frac{(n-1)}{n}\int\cos^{n-2}\theta\,d\theta,$$
and thus obtain the value of $\int\cos^3\theta\,d\theta$ and $\int\cos^4\theta\,d\theta$.

13. Prove that $$\int x^n e^x dx=x^n e^x-n\int x^{n-1}e^x dx,$$
and explain how this result may be used in evaluating such integrals as
$$\int x^3e^{2x}dx,\quad\int x^3e^{-2x}dx,\ \text{etc.}$$

14. Prove that $$\int x^{n-1}(\log x)^m dx = \int y^m e^{ny} dy,$$

where $x = e^y$, and explain how this result may be used in evaluating integrals such as

$$\int x^2(\log x)^3 dx, \quad \int x^{-2}(\log x)^3 dx.$$

15. Prove that

$$\int x^n \sin mx\, dx = -\frac{x^n}{m}\cos mx + \frac{n}{m}\int x^{n-1}\cos mx\, dx$$

$$= -\frac{x^n}{m}\cos mx + \frac{n}{m^2}x^{n-1}\sin mx - \frac{n \cdot \overline{n-1}}{m^2}\int x^{n-2}\sin mx\, dx,$$

and show how this may be used in evaluating such integrals.* Obtain a corresponding result in the case of

$$\int x^n \cos mx\, dx.$$

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* Examples 3, 4, 5 are cases of the use of the method of *Partial Fractions* in the integration of algebraic functions; 11-15, of the method of *Successive Reduction.* Cf. Lamb's *Infinitesimal Calculus*, §§ 80, 81.

CHAPTER VIII

THE DEFINITE INTEGRAL AND ITS APPLICATIONS

§ 53. **Introductory.**

In the last chapter we have considered the process of integration as the means of answering the question: What is the function whose differential coefficient is a given function? As we have already mentioned, there is another and a more important way of regarding the subject, in which integration appears as an operation of summation, or of finding the limit of the sum of a number of terms. We shall examine integration from this standpoint in the following sections.

§ 54. **Areas of Curves. The Definite Integral as an Expression for the Area.**

Let $y=f(x)$ be the equation of an ordinary continuous curve, and let us consider the area enclosed between the curve, the ordinates at $P_0(x_0, y_0)$ and $P(x, y)$, and the axis of x.

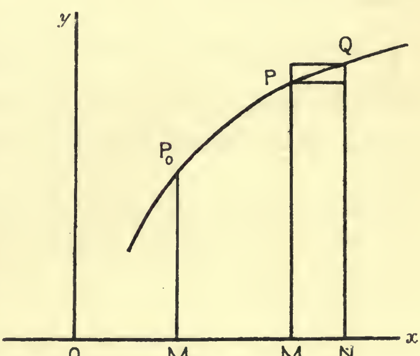

FIG. 22.

We assume, to begin with, that P_0P is above that axis.

This area is obviously a function of x, since to every position of P there corresponds a value of the area.

Let A stand for the area P_0M_0MP; $A+\delta A$ for the area P_0M_0NQ; and let Q be the point $(x+\delta x,\ y+\delta y)$. Then if the slope is positive from P to the point Q, we see by considering the inner and outer rectangles at P and the element of area there, that

$$y\,\delta x < \delta A < (y+\delta y)\,\delta x.$$

It follows that
$$y < \frac{\delta A}{\delta x} < y + \delta y.$$

If the slope is negative, the signs are reversed.

Hence in each case, when we let δx approach its limit zero, we have

$$\frac{dA}{dx} = y = f(x).$$

Thus if we write $F(x)$ for $\int f(x)\,dx$, and if C stands for an arbitrary constant, we have

$$A = F(x) + C.$$

Also, since A vanishes when $x = x_0$, $C = -F(x_0)$.

$$\therefore\ A = F(x) - F(x_0).$$

This expression
$$F(x) - F(x_0)$$

is an important one, and the symbol

$$\int_{x_0}^{x} f(x)\,dx$$

is used to denote it.

$\int_{x_0}^{x_1} f(x)\,dx$ *is called the definite integral of* $f(x)$ *with regard to* x *between the limits* x_0 *and* x_1, *and its value is obtained by subtracting the value of the indefinite integral* $\int f(x)\,dx$ *for* $x = x_0$ *from that for* $x = x_1$.

With this notation *the area of the curve* $y = f(x)$ *included between the ordinates at* (x_0, y_0) *and* (x_1, y_1), *the axis of* x *and the curve is equal to* $\int_{x_0}^{x_1} f(x)\,dx$.

It can be shown by a similar argument, or otherwise, that if the curve cuts the axis between the limits x_0 and x_1, the definite integral gives the algebraical sum of the areas, those above the axis of x being taken positive, those below the axis negative.

Ex. 1. To find the area of the sine curve

$$y = a \sin bx$$

from $x=0$ to $x=\frac{\pi}{2b}$.

$$\text{The required area} = \int_0^{\frac{\pi}{2b}} a \sin bx\, dx$$

$$= \left[-\frac{a}{b} \cos bx \right]_0^{\frac{\pi}{2b}}$$

$$= \frac{a}{b}.$$

This notation $[F(x)]_{x_0}^{x_1}$ for $F(x_1) - F(x_0)$ is useful in evaluating Definite Integrals.

2. To find the area of the curve

$$y = a \sin^2 bx$$

from $x=0$ to $x=\frac{\pi}{b}$.

$$\text{The required area} = \int_0^{\frac{\pi}{b}} a \sin^2 bx\, dx$$

$$= \frac{a}{2} \int_0^{\frac{\pi}{b}} (1 - \cos 2bx)\, dx$$

$$= \frac{a}{2} \left[x - \frac{\sin 2bx}{2b} \right]_0^{\frac{\pi}{b}}$$

$$= \frac{a\pi}{2b}.$$

3. To find the area of the part of the parobola $y^2 = 4ax$ cut off by the lines $x = x_0$ and $x = x_1$.

$$\text{Here the required area} = 2 \int_{x_0}^{x_1} \sqrt{4ax}\, dx$$

$$= 4\sqrt{a} \int_{x_0}^{x_1} \sqrt{x}\, dx$$

$$= 4\sqrt{a} \left[\frac{2}{3} x^{\frac{3}{2}} \right]_{x_0}^{x_1}$$

$$= \frac{8\sqrt{a}}{3} \left(x_1^{\frac{3}{2}} - x_0^{\frac{3}{2}} \right).$$

It follows that the area cut off by any ordinate P′NP is $\frac{2}{3}$ of the rectangle upon PP′ as base, with AN for its altitude.

4. To find the area of a circle of radius a.

Let the equation of the circle be

$$x^2 + y^2 = a^2.$$

$$\text{Then the required area} = 4 \int_0^a \sqrt{a^2 - x^2}\, dx.$$

This integral can be obtained by substituting

$x = a \sin \theta$. [Cf. p. 115.]

Or we may quote the result obtained above [§ 52]:

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Using this result, the area of the circle becomes

$$2\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right]_0^a:$$

i.e.

$$\pi a^2.$$

5. Prove that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

6. Prove that the area between the hyperbola

$$xy=c^2,$$

the axis of x, and the ordinates at (x_0, y_0), (x_1, y_1) is

$$c^2\log\left(\frac{x_1}{x_0}\right),$$

when x_0 and x_1 are both positive.

7. Prove the following:—

(i.) $$\int_0^{\frac{\pi}{4}}\frac{dx}{\cos x}=\log(\sqrt{2}+1)=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{\sin x}.$$

(ii.) $$\int_0^{\frac{\pi}{2}}\sin^2 x\,dx=\frac{\pi}{4}=\int_0^{\frac{\pi}{2}}\cos^2 x\,dx.$$

(iii.) $$\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\sin^2 x+b^2\cos^2 x}=\frac{\pi}{2ab}=\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}.$$

(iv.) $$\int_0^1\sin^{-1}x\,dx=\int_0^{\frac{\pi}{2}}\theta\cos\theta\,d\theta=\frac{\pi}{2}-1.$$

(v.) $$\int_0^a\frac{dx}{\sqrt{a-x}}=2\sqrt{a}.$$

(vi.) $$\int_1^2\frac{dx}{x\sqrt{x^2-1}}=\frac{\pi}{3}.$$

8. Prove that when m and n are positive integers

(i.) $$\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\,d\theta=\frac{2n-1\,.\,2n-3\ldots3\,.\,1}{2n\,.\,2n-2\ldots4\,.\,2}\,\frac{\pi}{2}=\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\,d\theta.$$

(ii.) $$\int_0^{\frac{\pi}{2}}\sin^{2n-1}\theta\,d\theta=\frac{2n-2\,.\,2n-4\ldots4\,.\,2}{2n-1\,.\,2n-3\ldots5\,.\,3}=\int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta\,d\theta.$$

(iii.) $$\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta\,d\theta=\frac{m-1}{m+n}\int_0^{\frac{\pi}{2}}\sin^{m-2}\theta\cos^n\theta\,d\theta.$$

(iv.) $$\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta\,d\theta=\frac{2}{15}.$$

(v.) $$\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^8\theta\,d\theta=\frac{5\,.\,3\,.\,1}{14\,.\,12\,.\,10}\int_0^{\frac{\pi}{2}}\cos^8\theta\,d\theta=\frac{5\pi}{212}.$$

In cases where integration is not possible there are various approximate methods of finding the area. The expressions for the area of a trapezium or a portion of a parabola give the trapezoidal and parabolic rules,* and we shall see more fully in §§ 55-56 how the inner and outer rectangles may be applied. The value of a definite integral may also be obtained by mechanical means by the use of different instruments, of which the planimeters are perhaps the best known.

Ex. Evaluate the following integrals by the trapezoidal method, *i.e.* find the sum of the inscribed trapeziums corresponding to the divisions named :—

(i.) $\int_1^{12} x^2 dx$, dividing the interval into 11 equal parts, and compare with the result of integration.

Answers, $577\frac{1}{2}$; $575\frac{2}{3}$.

(ii). $\int_{31^\circ}^{32^\circ} \cos x\, dx$, by dividing the interval into 6 equal parts, and compare as above.

Answers, ·0148; ·0149.

§ 55. The Definite Integral as the Limit of a Sum.

In the last article we have shown that the symbol $\int_{x_0}^{x_1} f(x)dx$ represents the area between the curve

$$y = f(x),$$

the axis of x, and the bounding ordinates. We shall now obtain an expression for this area as the *limit of a sum*, and thus see in what way the process of integration may be viewed as a summation.

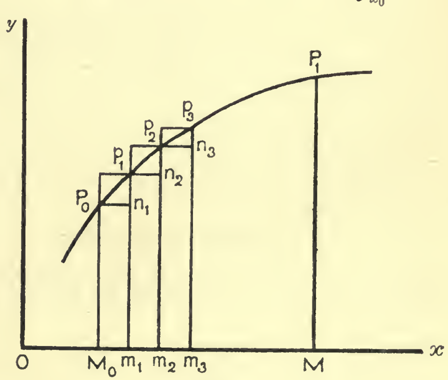

Fig. 23.

Let P_0P_1 be any portion of the curve in which the slope remains positive.

* Cf. Lamb's *Calculus*, § 112; Osgood's *Calculus*, p. 406; Gibson's *Calculus*, p. 329.

Divide the interval M_0M_1 into n equal parts δx, so that

$$n\delta x = x_1 - x_0:$$

erect the ordinates m_1p_1, m_2p_2, etc.; and construct inner and outer rectangles as in Fig. 23.

Then the difference between the sum of these outer rectangles and the sum of the inner rectangles is $(y_1 - y_0)\delta x$, and this may be made as small as we please by increasing the number of intervals and decreasing their size.

Also the area of the curve lies between these two sums. Therefore this area is the limit of either sum as δx approaches zero.

Now the sum of the inner set of rectangles

$$= [f(x_0)\delta x + f(x_0 + \delta x)\delta x + \ldots + f(x_0 + \overline{n-1}\,.\,\delta x)\delta x]$$

$$= \sum_{r=0}^{r=n-1} f(x_0 + r\delta x)\,\delta x.$$

But the area is $[F(x_1) - F(x_0)]$ where $F(x) = \int f(x)dx$, and we agreed to denote this by $\int_{x_0}^{x_1} f(x)dx$.

$$\therefore \int_{x_0}^{x_1} f(x)dx = \underset{\substack{\delta x \to 0 \\ n\delta x = (x_1 - x_0)}}{\text{Lt}} \sum_{r=0}^{r=n-1} f(x_0 + r\delta x)\,\delta x$$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_0}^{x_1} f(x)\delta x, \text{ written shortly.}$$

It is easy to remove the restriction placed upon $f(x)$ that the slope of the curve should be positive from P_0 to P_1; and to show that this result holds for any ordinary continuous curve whether it ascends or descends, and is above or below the axis in the interval x_0 to x_1.

It is only necessary to point out that in the case of such a portion of the curve $y = f(x)$ as is given in Fig. 24, the area of the portion of the curve marked II will appear as a negative area, and, if $\int f(x)dx = F(x)$,

$$\int_a^b f(x)dx, \quad \text{or} \quad [F(b) - F(a)],$$

is equal to

$$(I) - (II) + (III).$$

The great importance of the Integral Calculus depends upon the fact that many geometrical and physical quantities (*e.g.*

volumes and surfaces of solids, centres of gravity and pressure, total pressure, radius of gyration, etc.) may be expressed in terms of the limits of certain sums. The problem of obtaining these quantities is thus reduced to a question of integration.

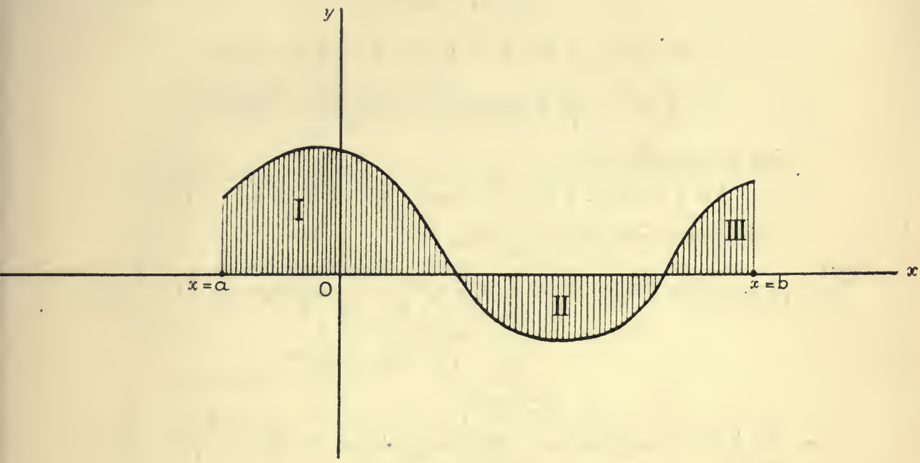

FIG. 24.

We have already remarked that the symbol of integration $\int$ really stands for the large S of summation, and it was in the attempts to calculate areas bounded by curves that the Integral Calculus was discovered.

It is also possible to begin the study of integration with the definition of the symbol

$$\int_{x_0}^{x_1} f(x)dx$$

as the limit of a sum, and to develop the whole subject from that definition.*

§ 56. **The Evaluation of a Definite Integral from its Definition as the Limit of a Sum.**

It is instructive to see how, by algebraical methods, the values of certain definite integrals may be obtained direct from this summation.

* Cf. Lamb's *Calculus*, §§ 90, 91.

For example, in the case of the parabola

$$y=x^2,$$

we can obtain the area, or the Definite Integral, as follows:—

$$\sum_{r=0}^{r=n-1}(x_0+r\,\delta x)^2\,\delta x$$

$$=\delta x[x_0^2+(x_0+\delta x)^2+(x_0+2\delta x)^2+(x_0+(n-1)\,\delta x)^2]$$

$$=\delta x\left(nx_0^2+n(n-1)x_0\delta x+\frac{(n-1)n(2n-1)}{6}(\delta x)^2\right),$$

using the results for

$$1+2+3+\ldots+(n-1),\quad\text{and}\quad 1^2+2^2+\ldots+(n-1)^2.$$

Therefore, since $n\,\delta x=(x_1-x_0)$,

$$\sum_{r=0}^{r=n-1}\delta x(x_0+r\,\delta x)^2=x_0^2(x_1-x_0)+x_0(x_1-x_0)^2\left(1-\frac{1}{n}\right)+\frac{1}{6}(x_1-x_0)^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right).$$

$$\therefore\quad \underset{\substack{\delta x\to 0\\ n\,\delta x=x_1-x_0}}{\mathrm{Lt}}\ \sum_{r=0}^{r=n-1}\delta x(x_0+r\,\delta x)^2$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\left[x_0^2(x_1-x_0)+x_0(x_1-x_0)^2\left(1-\frac{1}{n}\right)+\frac{1}{6}(x-x_0)^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$

$$=x_0^2(x_1-x_0)+x_0(x_1-x_0)^2+\frac{1}{3}(x_1-x_0)^3$$

$$=\frac{x_1^3-x_0^3}{3}.$$

$$\therefore\quad\int_{x_0}^{x_1}x^2dx=\frac{x_1^3-x_0^3}{3}.$$

Ex. Prove in the same way that

$$\int_0^{\frac{\pi}{2m}}\cos mx\,dx=\frac{1}{m},$$

and

$$\int_0^b e^{-ax}\,dx=\frac{1}{a}(1-e^{-ab}).$$

§ 57. Properties of $\int_{x_0}^{x_1}f(x)dx$.

The following properties of the Definite Integral may be deduced from either of the definitions of this symbol:—

I. $\int_{x_0}^{x_1}f(x)dx=-\int_{x_1}^{x_0}f(x)dx.$

II. $\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{\xi} f(x)\,dx + \int_{\xi}^{x_1} f(x)\,dx.$

III. *The integral of an even function between the limits* $-a$ *and* $+a=$ *twice the integral of the function between* 0 *and* a.

E.g.
$$\int_{-a}^{a} x^2dx = 2\int_0^a x^2dx = \frac{2}{3}a^3,$$
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \frac{\pi}{2}.$$

IV. *The integral of an odd function between the limits* $-a$ *and* $+a$ *is zero.*

E.g.
$$\int_{-a}^{+a} x^3dx = 0, \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3\theta\,d\theta = 0.$$

Similarly
$$\int_0^{\pi} \sin^m\theta\cos^{2n+1}\theta\,d\theta = 0,$$
m, n being positive integers.

V. In applying the method of the "change of the variable" to the evaluation of definite integrals, we need not express the result in terms of the original variable. We need only give the new variable the values at its limits which correspond to the change from x_0 to x_1 in the variable x,* care being taken in the case of a many-valued function that the values we thus allot are those which correspond to the given change in x.

E.g.
$$\int_0^a \sqrt{a^2-x^2}\,dx$$
$$= a^2\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta, \quad \text{putting } x = a\sin\theta,$$
$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)d\theta$$
$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{2}} = \pi\frac{a^2}{4}.$$

* If we integrate $\int_0^{\pi} \sin x dx$ by the substitution $\sin x = y$, it appears at first that we obtain zero instead of 2 for the result. It is not hard to trace the reason for this discrepancy, and this example shows that in the use of this method particular care has to be taken.

GEOMETRICAL AND PHYSICAL APPLICATIONS OF INTEGRATION

§ 58. **Application to Fluid Pressure.**

The determination of the pressure of a liquid upon a plane surface shows more clearly than many other examples the real meaning of integration.

Let us take the simplest possible case. Suppose we have a rectangular trough, resting upon a horizontal plane, and that we fill it full of water. There is a certain force pressing out the ends of the tank due to the weight of the water. We shall find the amount of this force for one end: in other words, the *whole pressure* of the water upon an end of the trough.

The *whole pressure* is made up of the pressures distributed over the surface considered, and another problem is to find where the resultant pressure acts. The point at which it acts is called the *Centre of Pressure.*

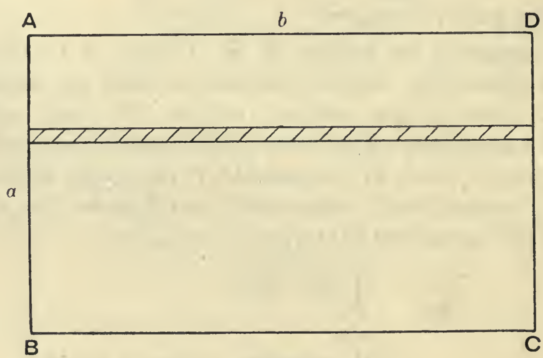

FIG. 25.

Let the rectangular section be of breadth b ft. and depth a ft. (Cf. Fig. 25.) We take the line AD, which lies in the surface of the water, as the axis of y, and the vertical line AB as the axis of x.

We learn from Physics that the pressure per sq. ft. at a depth x ft. below the surface of the water is wx lbs., w being the weight in lbs. of a cubic ft. of water. [1 cub. ft. of water weighs 1000 oz. or $62\frac{1}{4}$ lbs.]

Suppose that the rectangle (cf. Fig. 25) is divided into n equal strips by lines parallel to the axis of y, the thickness of each strip being δx.

Let $x_1, x_2, x_3, \dots x_n$ be the distances from A to the points where the *lower* edges of the strips cut the axis of x.

Let $\delta P_r =$ the pressure on the r^{th} strip.

Then we have

$$\begin{aligned} wx_1\, b\,\delta x &> \delta P_1 > 0, \\ wx_2\, b\,\delta x &> \delta P_2 > wx_1\, b\,\delta x, \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \\ wx_n\, b\,\delta x &> \delta P_n > wx_{n-1}\, b\,\delta x, \end{aligned}$$

since the actual pressure on each strip is *greater* than what we would obtain if we were to take the pressure as uniform over it, and equal to that at its upper edge: also it is *less* than what we would obtain if we were to take the pressure as uniform over it, and equal to that at its lower edge.

Now the difference between the sum of the numbers in the first column and those in the third is equal to

$$wx_n\, b\,\delta x \quad \text{or} \quad w\,\delta x \times \text{the area of the rectangle.}$$

It follows that when $\delta x \to 0$, $n\,\delta x$ remaining always equal to a, the two sums become equal in the limit.

But the actual pressure,

$$\delta P_1 + \delta P_2 + \dots + \delta P_n,$$

lies between these two sums.

It follows that this pressure, which we shall denote by P, is equal to the limit when $\delta x \to 0$ of either sum.

In other words,

$$P = \underset{\substack{\delta x \to 0 \\ n\,\delta x = a}}{\text{Lt}} \sum_{r=0}^{n-1} wx_r \,.\, b\,\delta x = \int_0^a wx \,.\, b\,dx = \tfrac{1}{2} wa^2 b.$$

It will be noticed that the pressure is equal to *the area of the surface immersed multiplied by the pressure at its Centre of Gravity*, a theorem which can be shown to be true in general.

We take another example, where the section is not rectangular. Let the section of the trough be a semi-circle, the diameter lying in the surface of the water, its length being $2a$.

In this case it is convenient to take the origin at the centre, and the axes of x and y vertical and horizontal.

As before, we divide the section into strips parallel to the axis of y, the breadth of each strip being δx.

Let QNQ′ be the upper edge of one of these strips. Let ON $= x$ and QN $= y$. [Cf. Fig. 26.]

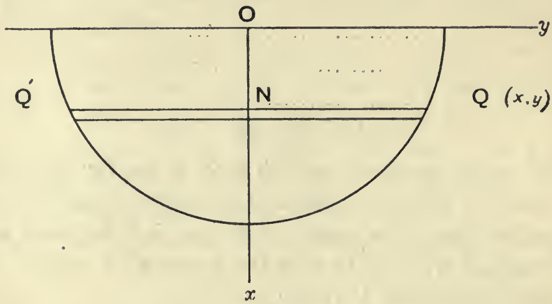

FIG. 26.

Let δP = the pressure on this strip.

Then we have

$$2w(x+\delta x)y\,\delta x > \delta P > 2wx(y+\delta y)\,\delta x.$$

If we were to add the terms in the first and third columns obtained in this way, we would find that the limit of each sum would be the same, when $\delta x \to 0$, $n\,\delta x$ remaining equal to a.

Also the pressure P lies between the two sums.

It follows that P is equal to the limit of either sum, and in finding this limit we can omit the terms involving $\delta x\,\delta y$. The sum of these terms vanishes in the limit.

Thus $$P = 2w\int_0^a xy\,dx, \text{ where } y = \sqrt{a^2 - x^2},$$

$$= 2w\int_0^a x\sqrt{a^2 - x^2}\,dx$$

$$= 2w\left[-\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a$$

$$= \tfrac{2}{3}wa^3.$$

When we have obtained the position of the C.G. of a semicircle (cf. p. 124), we shall see that the above answer agrees with the general theorem to which we have referred.

These results can easily be extended, and a general formula obtained. However the student is advised, at this stage, to work out such examples from first principles. When he has grasped the meaning of the argument used in the above discussion, it is unnecessary to write down the inequalities on which it depends in full. For example, in the case of the semi-circle, it would be sufficient to say that

$$\delta P = 2wx \,.\, y\delta x, \text{ to the first order};$$

so that on integrating over the semi-circle

$$P = 2w\int_0^a xy\,dx.$$

Ex. 1. A water main 6 ft. in diameter is just full of water. Show that the pressure on the gate that closes the main is over $2\frac{1}{4}$ tons.

2. A vertical masonry dam is in the form of a rectangle 200 ft. long at the surface of the water, and 50 ft. deep. Show that when full it has to withstand a pressure of nearly 7000 tons.

3. The bank of a reservoir is inclined at an angle of 60° to the horizontal. If the depth of the water is 30 ft., show that the normal pressure on the section 100 ft. long is over 1400 tons.

§ 59. **Application to Areas in Polar Co-ordinates.**

When the equation of the curve is given in polar co-ordinates, the area of the sector bounded by $\theta=\theta_0$ and $\theta=\theta_1$ can be shown to be

$$\underset{\delta\theta\to 0}{\mathrm{Lt}} \sum_{\theta=\theta_0}^{\theta=\theta_1}\left(\frac{1}{2}r^2\delta\theta\right),$$

with the same notation as before. Hence if the curve is $r=f(\theta)$, the sectorial area is

$$\frac{1}{2}\int_{\theta_0}^{\theta_1}[f(\theta)]^2 d\theta.$$

Polar co-ordinates are often used in finding the area of a loop of a curve.

For example, the lemniscate

$$r^2 = a^2\cos 2\theta$$

has a loop between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$.

The area of this loop $=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\theta$

$$=\frac{a^2}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta.$$

$\therefore$ the area of the loop $=a^2\int_0^{\frac{\pi}{4}} \cos 2\theta d\theta$ (Cf. § 57, III.)

$$=a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{4}}$$

$$=\frac{a^2}{2}.$$

Similarly, in the Folium of Descartes, whose equation is

$$x^3+y^3=3axy,$$

there is a loop in the first quadrant.

Using polar co-ordinates we find that the area of the loop

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}} \left\{\frac{3a\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}\right\}^2 d\theta$$

$$=\frac{9}{2}a^2\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2} d\theta$$

$$=\frac{9}{2}a^2\int_0^{\infty} \frac{t^2}{(1+t^3)^2} dt, \quad \text{putting } \tan\theta=t, \quad \text{(Cf. § 57, V.)}$$

$$=\frac{3}{2}a^2\int_0^{\infty} \frac{d(t^3)}{(1+t^3)^2}$$

$$=\frac{3}{2}a^2\left(-\frac{1}{1+t^3}\right)_0^{\infty}$$

$$=\frac{3a^2}{2}.$$

Ex. Prove that the area of the cardioide $r=a(1-\cos\theta)$ is $\frac{3}{2}\pi a^2$.

§ 60. Application to Lengths of Curves.

The length of an arc P_0P_1 of the curve $y=f(x)$ may be regarded as the limit of the sum of the different chords into which P_0P_1 is divided by the ordinates at m_1, m_2, ... (cf. Fig. 23).

Hence

$$\text{arc } P_0P_1 = \underset{\delta x \to 0}{\text{Lt}} \sum_{x=x_0}^{x=x_1} \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x=x_0}^{x=x_1} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

$$= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{y_0}^{y_1} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy,$$

since $\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$ will differ from $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ by a very small quantity when δx is very small, and the sum of these differences multiplied by δx will vanish in the limit.

If polar co-ordinates are used, we obtain the two expressions

$$\int_{r_0}^{r_1} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\,dr \quad \text{and} \quad \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta,$$

since the chord is $\sqrt{(\delta r)^2 + (r\,\delta\theta)^2}$ to the first order.

Owing to the presence of the radical sign under the sign of integration, the problem of finding the length of the curve has been solved in only a limited number of cases.

Ex. 1. Prove that the length of the arc of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is equal to $a[\sqrt{2} + \log(\sqrt{2} + 1)]$.

2. Prove that the length of the cardioide $r = a(1 - \cos\theta)$ is $8a$.

§ 61. The Volume of a Solid of Revolution.

Let the solid be formed by the revolution of the curve

$$y = f(x)$$

about the axis of x.

We wish to obtain the volume contained between the planes $x = x_0$ and $x = x_1$.

We suppose the interval x_0 to x_1 divided up into n equal parts δx, as in § 55; and we take the sections of the solid by the planes perpendicular to the axis at these points.*

---

* We might also proceed as follows :—

We have $$\pi y^2 \delta x \lesseqgtr \delta V \lesseqgtr \pi (y + \delta y)^2 \delta x,$$

V standing for the volume up to the section considered, and δV for the increment of volume.

Thus $$\frac{dV}{dx} = \pi y^2.$$

The rest of the argument presents no difficulty.

If we let inner and outer discs take the place of the inner and outer rectangles of our former argument, it readily follows that the required volume is given by

$$\operatorname*{Lt}_{\substack{\delta x \to 0 \\ n\,\delta x = x_1 - x_0}} \sum_{r=0}^{r=x-1} \pi {y_r}^2 \delta x,$$

where $$y_r = f(x_0 + r\,\delta x).$$

Thus the volume $= \pi \int_{x_0}^{x_1} y^2 dx = \pi \int_{x_0}^{x_1} [f(x)]^2 dx.$

When the axis of y is the axis of revolution, the area of the section is πx^2 instead of πy^2, and the volume becomes

$$\pi \int_{y_0}^{y_1} x^2 dy.$$

Ex. 1. The portion of the parabola $y^2 = 4ax$ from the vertex to the point $P(x, y)$ revolves about Ox. Prove that the volume of the cup we thus obtain is $2a\pi x^2$.

2. Obtain the volume of a sphere by considering the rotation of the semicircle $x^2 + y^2 = a^2$ about Ox.

3. Find the volume (i.) of a right circular cone, and (ii.) of a cone in which the base is any plane figure of area A, and the perpendicular from the vertex upon the base is h.

4. Prove that the volume of a spherical cap of height h is $\pi h^2\left(r - \frac{h}{3}\right)$, where r is the radius of the sphere.

§ 62. **The Surface of a Solid of Revolution.**

It is easy to show that the surface of a right circular cone whose vertical angle is 2α and whose generators are of length l is $\pi l^2 \sin \alpha$. We can deduce from this that the surface of the slice of a cone obtained by revolving a line PQ about Ox is equal to

$$2\pi \,.\, PQ \,.\, NR,$$

where NR is the ordinate from the middle point of PQ.

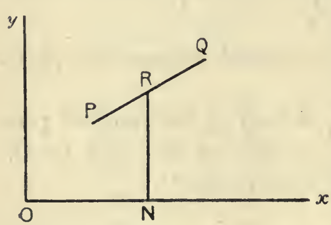

FIG. 27.

Suppose then that an arc P_0P_1 of the curve $y = f(x)$ rotates about Ox. The area of the surface generated by P_0P_1 is the limiting value of the sum of the areas of the surfaces generated by the chords into which we suppose this arc divided. Thus the area of the surface generated by P_0P_1

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \sum_{x=x_0}^{x=x_1} 2\pi\left(y + \frac{1}{2}\delta y\right)\sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

$$= 2\pi\int_{x_0}^{x_1} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx, \text{ where } y = f(x).$$

This may be written $2\pi\int_{s_0}^{s_1} y\,ds$, by changing the variable from x to s, where s is the length of the arc from a fixed point to the point (x, y).

When the axis of revolution is the axis of y, we obtain in the same way the expression $2\pi\int_{s_0}^{s_1} x\,ds$.

Ex. 1. Obtain the expression for the surface of a sphere of radius a.

Here we take the curve $y = \sqrt{a^2 - x^2}$,

$$\text{and the surface} = 4\pi\int_0^a \sqrt{a^2 - x^2}\sqrt{1 + \frac{x^2}{y^2}}\,dx$$

$$= 4\pi a\int_0^a dx$$

$$= 4\pi a^2.$$

2. Prove that the area of the portion of a sphere cut off by two parallel planes is equal to the area which they cut off from the circumscribing cylinder whose generators are perpendicular to these planes.

3. Prove that the area of the surface formed by rotating the circle of radius a, whose centre is distant d from the axis of x, about that axis, is $4\pi^2 ad$.

§ 63. The Centre of Gravity of a Solid Body.

If a number of particles of masses m_1, m_2, ... are situated at the points (x_1, y_1, z_1), ..., their C.G. is given by

$$\bar{x} = \frac{\Sigma(m_r x_r)}{\Sigma(m_r)},\quad \bar{y} = \frac{\Sigma(m_r y_r)}{\Sigma(m_r)},\quad \bar{z} = \frac{\Sigma(m_r z_r)}{\Sigma(m_r)}.$$

Now we may suppose any solid body broken up into small elements of mass. Let (x, y, z) be the C.G. of the element δm. Then we may write these results for a solid body in the form

$$\bar{x} = \frac{\underset{\delta m \to 0}{\mathrm{Lt}}\ \Sigma x\,\delta m}{\mathrm{M}},\quad \bar{y} = \frac{\underset{\delta m \to 0}{\mathrm{Lt}}\ \Sigma y\,\delta m}{\mathrm{M}},\quad \bar{z} = \frac{\underset{\delta m \to 0}{\mathrm{Lt}}\ \Sigma z\,\delta m}{\mathrm{M}}.$$

In many cases we can transform these expressions into integrals which we can evaluate by the methods already employed, though in general they involve integration with regard to more than one variable, and such integrals cannot be discussed here.

We add some illustrative examples :—

Ex. 1. **The Centre of Gravity of a Semi-circular Plate.**

Take the boundary of the plate along the axis of y, and suppose the semicircle divided by a set of lines parallel to that axis and very near one another. The C.G. of each of these strips PQ′ lies on the axis of x, and therefore the C.G. of the semicircle lies on Ox.

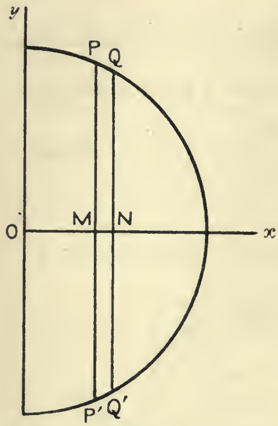

Fig. 28.

We thus have

$$\bar{x} = \frac{\underset{\delta m \to 0}{\text{Lt}} \; \Sigma x\,\delta m}{M}$$

$$= \frac{2\int_0^a xy\,dx}{\frac{\pi a^2}{2}}$$

$$= \frac{4}{\pi a^2}\int_0^a x\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4}{\pi a^2}\left[-\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}\right]_0^a$$

$$= \frac{4a}{3\pi};$$

and $\bar{y} = 0$.

2. **The Centre of Gravity of a uniform Solid Hemisphere.**

Let the axis of x be the radius to the pole of the hemisphere, and suppose the solid divided up into thin slices by a set of planes perpendicular to this axis.

Then the C.G. of each of these slices lies on this axis, and therefore the C.G. of the hemisphere does so also.

Also we have

$$\bar{x} = \frac{\int_0^a \pi x y^2\,dx}{\frac{2}{3}\pi a^3}$$

$$= \frac{3}{2a^3}\int_0^a x(a^2 - x^2)\,dx$$

$$= \frac{3}{2a^3}\left(a^2\int_0^a x\,dx - \int_0^a x^3\,dx\right)$$

$$= \frac{3}{2a^3}\left(a^2\left[\frac{x^2}{2}\right]_0^a - \left[\frac{x^4}{4}\right]_0^a\right).$$

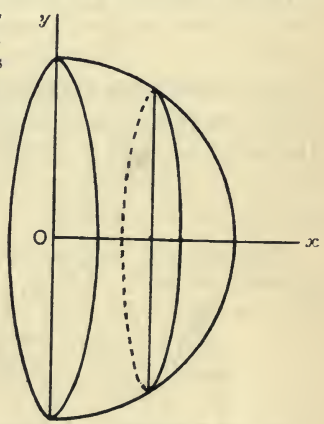

Fig. 29.

$$\therefore\ \bar{x} = \frac{3}{2}a\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{3}{8}a.$$

3. Prove that the C.G. of any cone or pyramid upon a plane base is one fourth of the way up the line from the vertex to the C.G. of the base.

4. Prove that the C.G. of the upper portion of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is at the point $\left(0, \frac{4b}{3\pi}\right)$.

§ 64. **Moments of Inertia.**

The moment of inertia, I, of a set of particles $m_1, m_2, \ldots$ with respect to an axis from which they are distant r_1, r_2, etc., is the expression

$$m_1r_1^2+m_2r_2^2+\ldots,$$

and in the case of a continuous solid body we may express this as

$$I=\underset{\delta m \to 0}{\mathrm{Lt}}\ \Sigma r^2\delta m.$$

The radius of gyration k is defined by the equation

$$I=Mk^2.$$

In many cases we can obtain the values of I and k^2 by the use of the methods of integration we have been discussing.

We add some illustrative examples:—

Ex. 1. To find the radius of gyration of a thin rod of mass M and length $2l$, about an axis at right angles to the rod and passing through its centre.

Here $I=\underset{\delta m \to 0}{\mathrm{Lt}}\ \Sigma x^2\delta m$

$=\rho\int_{-l}^{l} x^2dx,$

where $2l\rho=M$,

$=2\rho\int_0^l x^2dx$

$=\frac{2}{3}\rho l^3$

$=\frac{Ml^2}{3}.$

$\therefore\ k=\frac{1}{3}l\sqrt{3}.$

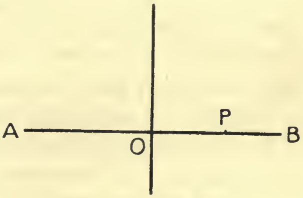

FIG. 30.

2. To find the moment of inertia of a solid circular cylinder about its axis.

We have $$I=\underset{\delta m \to 0}{\mathrm{Lt}}\ \Sigma r^2\delta m,$$

where $$\delta m=\rho h\{\pi(r+\delta r)^2-\pi r^2\}$$
$$=\pi\rho h\{2r\,\delta r+(\delta r)^2\},$$

ρ being the volume density, and h the height of the cylinder.

Therefore
$$I=\pi\rho h\int_0^a r^2 2r\,dr$$
$$=2\pi\rho h\int^a r^3\,dr$$
$$=\frac{\pi}{2}\rho h a^4.$$

But
$$\pi\rho h a^2=M.$$
$$\therefore\ I=M\frac{a^2}{2}.$$

3. Prove that the radius of gyration of a thin circular plate of radius a about a diameter as axis is $\frac{1}{4}a^2$.

EXAMPLES ON CHAPTER VIII

1. Find the areas bounded by

(i.) $y=\sin 2x,\ x=0,\ x=\frac{\pi}{2}$.

(ii.) $y=e^{-x}\sin 2x,\ x=0,\ x=\frac{\pi}{2}$.

(iii.) The hyperbola $xy=a^2,\ x=x_1,\ x=x_2$.

(iv.) $y=x^3,\ x=0,\ x=4$.

(v.) $y=2x^3$, the axis of y, and the lines $y=2$ and $y=4$.

2. Find the area of the part of the parabola $y=x^2-3x+2$ cut off by the x axis. What does $\int_0^2 y\,dx$ here represent?

3. Trace the parabola $(y-x-3)^2=x+y$, and find the area of the part of the curve cut off by the lines $x=0$ and $x=4\frac{1}{2}$.

4. Find the areas in polar co-ordinates of

(i.) The part of $r=a\theta$ included between $\theta=0$ and $\theta=2\pi$.

(ii.) A loop of each of the curves $r=a\sin 2\theta$, $a\sin 3\theta$, etc.

(iii.) A loop of each of the curves $r=a\cos 2\theta$, $a\cos 3\theta$, etc.

(iv.) The part of the hyperbola $r^2\sin\theta\cos\theta=a^2$ included between $\theta=\theta_1$ and $\theta=\theta_2$.

(v.) A sector of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the centre being the pole.

(vi.) Prove that the area between the two parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16a^2}{3}$.

(vii.) Prove that the area between the two ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ is $4ab\tan^{-1}\frac{b}{a}$.

5. By substituting $x=a\cos\theta$, $y=b\sin\theta$, show that the perimeter of the ellipse of semiaxes a, b is given by $4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\theta}\,.\,d\theta$, and deduce that for an ellipse of small eccentricity the perimeter is approximately $2\pi a\left(1-\frac{e^2}{4}\right)$.

6. Find the lengths of the following curves :—

(i.) The equiangular spiral $r=ae^{\theta\cot\alpha}$ from $\theta=0$ to $\theta=2\pi$.

(ii.) The spiral of Archimedes $r=a\theta$ from $\theta=0$ to $\theta=2\pi$.

(iii.) The catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ from $x=0$ to $x=a$.

(iv.) And show that the length of a complete undulation of the curve

$$y=b\sin\frac{x}{a}$$

is equal to the perimeter of an ellipse whose axes are $2\sqrt{a^2+b^2}$ and $2a$.

7. Find the volumes of the following solids :—

(i.) The solid formed by revolving the part of the line $x+y=1$ cut off by the axes, about the axis of x, and verify your result by finding the volume of the cone in the usual way.

(ii.) The spheroid formed by rotating the ellipse $9x^2+16y^2=144$ about the axis of x.

(iii.) The cup formed by the revolution of a quadrant of a circle about the tangent at the end of one of its bounding radii.

(iv.) The cup of height h formed by the revolution of the curve $a^2y=x^3$ about the axis of y.

(v.) The ring formed by the revolution of the circle $(x-a)^2+y^2=b^2$ about the axis of y.

(vi.) The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

And show that if S_0, S_1, S_2 are the areas of three parallel sections of a sphere at equal distances a, the volume included between S_0, S_2 and the spherical boundary is $\frac{a}{3}(S_0+4S_1+S_2)$.

8. The ellipse whose eccentricity is e rotates about its major axis. Prove that the area of the surface of the prolate spheroid thus formed is

$$2\pi b\left(b+\frac{a}{e}\sin^{-1}e\right).$$

9. The catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ rotates about the axis of y; prove that the area of the surface of the cup formed by the part of the curve from $x=0$ to $x=a$ is $2\pi a^2\left(1-\frac{1}{e}\right)$.

10. The cardioide $r=a(1-\cos\theta)$ revolves about the initial line; prove that the surface of the solid thus formed is $\frac{32}{5}\pi a^2$.

11. Find the C.G. of the following:—

(i.) A thin straight rod of length l in which the density varies as the distance from one end.

(ii.) An arc of a circle of radius a which subtends an angle 2α at the centre.

(iii.) A quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(iv.) A circular sector as in (ii.).

(v.) The segment of the sector of (iv.) bounded by the arc and its chord.

(vi.) A thin hemispherical shell of radius a.

12. Find the moments of inertia of each of the following:—

(i.) A thin straight rod, about an axis through an end, perpendicular to its length.

(ii.) A fine circular wire of radius a, about a diameter.

(iii.) A circular disc of radius a, about an axis through its centre perpendicular to the plane of the disc.

(iv.) A hollow circular cylinder of radii a, b, and height h, about its axis.

(v.) A sphere of radius a, about a tangent line.

(vi.) (α) A rectangle whose sides are $2a$ and $2b$, about an axis through its centre in its plane perpendicular to the side $2a$;

(β) about an axis through its centre perpendicular to its plane.

(vii.) An ellipse whose axes are $2a$ and $2b$,

(α) about the major axis a;

(β) about the minor axis b;

(γ) about an axis perpendicular to its plane through the centre.

N.B.—The case of the circle follows on putting $a=b$.

(viii.) An ellipsoid, semiaxes a, b, c, about the axis a.

N.B.—For the sphere $a=b=c$.

(ix.) A right solid whose sides are $2a$, $2b$, $2c$, about an axis through its centre perpendicular to the plane containing the sides b and c.

N.B.—Routh's Rule for these last four important cases can be easily remembered:—

$$\left.\begin{array}{c}\textit{Moment of Inertia about an axis}\\ \textit{of symmetry}\end{array}\right\} = \textit{mass}\,\frac{\left(\begin{array}{c}\textit{sum of squares of perpendicular}\\ \textit{semiaxes}\end{array}\right)}{3,\ 4,\ \textit{or}\ 5}.$$

The denominator is to be 3, 4, *or* 5 *according as the body is rectangular, elliptical, or ellipsoidal.*

Cf. Routh's *Rigid Dynamics*, vol. i. p. 6.

APPENDIX

Alternative Proofs for the Differentiation of x^n, e^x and $\log x$.

I. *The Differential Coefficient of* x^n.

Let $$y = x^n.$$

Then $$y + \delta y = (x + \delta x)^n$$
$$= x^n\left(1 + \frac{\delta x}{x}\right)^n.$$

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{x^n\left(1 + \frac{\delta x}{x}\right)^n - x^n}{\delta x}.$$

But by the Binomial Theorem, when $h < 1$,

$$(1 + h)^n = 1 + nh + \frac{n \,.\, n-1}{1 \,.\, 2} h^2 + \dots .$$

Therefore

$$\frac{\delta y}{\delta x} = \frac{x^n\left(1 + \frac{n}{x}\delta x + \frac{n \,.\, n-1}{1 \,.\, 2} \frac{(\delta x)^2}{x^2} + \dots\right) - x^n}{\delta x}.$$

$$\therefore \quad \frac{\delta y}{\delta x} = nx^{n-1} + \frac{n \,.\, n-1}{1 \,.\, 2} x^{n-2} \delta x + \dots *$$

provided that δx is so small that

$$-1 < \frac{\delta x}{x} < 1.$$

---

* The fact that we have an infinite series on the right hand sometimes causes difficulty to the student, as he imagines that what he calls the summing of the nfinite number of small terms involving δx, $(\delta x)^2$, etc. ... may give rise to a finite sum. The answer to this difficulty in general is to be found in a true view of the meaning of a convergent infinite series, but in the series here referred to we are able to say what the possible error can be if we stop after a certain number of terms. We thus exclude the infinite series from our argument.

Hence $$\underset{\delta x \to 0}{\text{Lt}}\left(\frac{\delta y}{\delta x}\right) = nx^{n-1},$$
and the differential coefficient of x^n is nx^{n-1}.

II. *To differentiate e^x.*

Let $$y = e^x.$$

Then $$y + \delta y = e^{x+\delta x}.$$

$$\therefore\ \delta y = e^{x+\delta x} - e^x$$
$$= e^x(e^{\delta x} - 1)$$
$$= e^x\left(1 + \frac{\delta x}{1} + \frac{(\delta x)^2}{2\,!} + \ldots - 1\right).$$

$$\therefore\ \frac{\delta y}{\delta x} = e^x\left(1 + \frac{\delta x}{2\,!} + \frac{(\delta x)^2}{3\,!} + \ldots\right).$$

Proceeding to the limit, $\dfrac{dy}{dx} = e^x.$

Thus $$\frac{d}{dx}(e^x) = e^x.$$

III. *To differentiate $\log x$.*

Let $$y = \log x.$$

Then $$y + \delta y = \log(x + \delta x).$$

Therefore $$\delta y = \log(x + \delta x) - \log x$$
$$= \log\left(1 + \frac{\delta x}{x}\right)$$
$$= \frac{\delta x}{x} - \frac{1}{2}\left(\frac{\delta x}{x}\right)^2 + \frac{1}{3}\left(\frac{\delta x}{x}\right)^3 - \ldots, \text{ if } \left|\frac{\delta x}{x}\right| < 1.$$

$$\therefore\ \frac{\delta y}{\delta x} = \frac{1}{x} - \frac{1}{2x}\left(\frac{\delta x}{x}\right) + \frac{1}{3x}\left(\frac{\delta x}{x}\right)^2 - \ldots.$$

Proceeding to the limit, $\dfrac{dy}{dx} = \dfrac{1}{x}.$

Thus $$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

ANSWERS

CHAPTER I. (p. 15)

1. (i.) $x^2+y^2=\frac{c^2-2a^2}{2}$. (ii.) $x=\frac{c}{4a}$.

 (iii.) $x^4+2x^2y^2+y^4+2a^2(y^2-x^2)+a^4-c^4=0$.

2. $x+4y-11=0$. 3. $\left(-\frac{13}{11}, \frac{19}{11}\right)$.

4. The parallel lines through O are

$$3x-2y=0,\quad 4x+y=0,\quad 19x+13y=0.$$

 The perpendicular lines through O are

$$2x+3y=0,\quad x-4y=0,\quad 13x-19y=0.$$

 The parallels through (2, 2) are

$$3x-2y=2,\quad 4x+y=10,\quad 19x+13y=64.$$

 The perpendiculars through (2, 2) are

$$2x+3y=10,\quad x-4y+6=0,\quad 13x-19y+12=0.$$

5. $x+3y-7=0$.

6. $7x+7y-36=0$ is the bisector of the acute angle.
 $x-y-12=0$ is the bisector of the obtuse angle.

7. (i.) (1, 2), (3, 4), (5, 3). (ii.) $\frac{3}{5}$, -3, $\frac{6}{7}$.

 (iii.) The internal bisectors are

$$\frac{x-y+1}{\sqrt{2}}=\frac{-x+4y-7}{\sqrt{17}},\quad \frac{x-y+1}{\sqrt{2}}=\frac{-x-2y+11}{\sqrt{5}},\quad \frac{x-4y+7}{\sqrt{17}}=\frac{x+2y-11}{\sqrt{5}}.$$

 The external bisectors are

$$\frac{x-y+1}{\sqrt{2}}=\frac{x-4y+7}{\sqrt{17}},\quad \frac{x-y+1}{\sqrt{2}}=\frac{x+2y-11}{\sqrt{5}},\quad \frac{x-4y+7}{\sqrt{17}}=\frac{-x-2y+11}{\sqrt{5}}.$$

8. If the points (0, 0), (2, 4), (-6, 8) be called A, B, C respectively,

 (i.) BC is $x+2y-10=0$,
 CA is $4x+3y=0$,
 AB is $2x-y=0$.

 (ii.) $\tan A=2$, $\tan B=\infty$, $\tan C=\frac{1}{2}$.

(iii.) Median through A is $y+3x=0$,
Median through B is $y-4=0$,
Median through C is $6x+7y-20=0$.

(iv.) The perpendicular from A on BC is the line AB; its length is $2\sqrt{5}$.
The perpendicular from B on CA is the line $3x-4y+11=0$; its length is 4.
The perpendicular from C on AB is the line CB; its length is $4\sqrt{5}$.

(v.) $x+2y=0$,
$4x+3y-20=0$,
$2x-y+20=0$.

(vi.) $x=-\frac{4}{3}$, $y=4$.

(vii.) $\left(\frac{-2(3-\sqrt{5})}{3+\sqrt{5}}, \frac{2(4+\sqrt{5})}{3+\sqrt{5}}\right)$, $(-3, 4)$, $\left(-\frac{1}{2}, 4\right)$.

9. $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)$.

10. (i.) 6. (ii.) 40.

CHAPTER II. (p. 29)

4. $y-9x+16=0$.

5. $x=\frac{1}{4}$, $y=\frac{1}{2}: x=1$, $y=-1$.

6. $u-gt$; $-g$.

7. $2\pi rh\,\delta r$.

8. $\delta V=4\pi r^2\delta r$, 50·27, 502·66.

9. $(a+2bt)$, $\delta l=(a+2bt)\,\delta t$.

12. $\frac{1}{\sqrt{3}}$ feet per second.

CHAPTER III. (p. 43)

1. (i.) $\frac{dy}{dx}=\frac{3(x-1)^2(x+1)}{2x^{\frac{5}{2}}}$.

(ii.) $\frac{dy}{dx}=\frac{a-x}{\sqrt{2ax-x^2}}$.

(iii.) $\frac{dy}{dx}=\frac{2x+3}{2\sqrt{(x+1)(x+2)}}$.

(iv.) $\frac{dy}{dx}=(x+a)^{p-1}(x+b)^{q-1}\{(p+q)x+qa+pb\}$.

(v.) $\frac{dy}{dx}=\frac{1}{(1-x)\sqrt{1-x^2}}$.

(vi.) $\frac{(a-x)^{p-1}}{(b-x)^{q+1}}(qa-pb+(p-q)x)$.

(vii.) $x^{n-1}\frac{(b(n-m)x^m+na)}{(bx^m+a)^2}$.

(viii.) $mnx^{n-1}(1+x^n)^{m-1}$.

(ix.) $\frac{x(\sqrt{x^2+a^2}+\sqrt{x^2-a^2})}{\sqrt{x^4-a^4}}$.

(x.) $-x\left\{(x^2+a^2)^{-\frac{3}{2}}+(x^2-a^2)^{-\frac{3}{2}}\right\}$.

(xi.) $\frac{3x^2}{(1-x^2)^{\frac{5}{2}}}$.

(xii.) $\frac{1-x^2}{(1+x+x^2)^{\frac{1}{2}}(1-x+x^2)^{\frac{3}{2}}}$.

2. (i.) $\frac{2a}{y_0}$. (ii.) $-\frac{x_0}{y_0}$. (iii.) $\mp\frac{b^2x_0}{a^2y_0}$. (iv.) $-\frac{y_0}{x_0}$.

4. 7·96 miles per hour. 5. 8 miles per hour ; 4 miles per hour.

8. $\frac{dp}{dv} = -\frac{\gamma p}{v}$.

9. When the pressure decreases, the volume increases, and conversely.

CHAPTER IV. (p. 59)

1. (i.) $3 \sin x \cos x (\sin x - \cos x)$. (ii.) $\sec^4 x$. (iii.) $\frac{4 \sin x}{\cos^3 x}$.
(iv.) $\frac{-4 \cos x}{\sin^3 x}$. (v.) $\frac{2 \cos x}{(1 - \sin x)^2}$. (vi.) $\frac{2 \sin x}{(1 + \cos x)^2}$.

3. (i.) $x^{m-1}[m \sin (x^n) + nx^n \cos (x^n)]$.
(ii.) $x^{m-1}[m \cos (x^n) - nx^n \sin (x^n)]$.
(iii.) $x^{m-1}[m \tan (x^n) + nx^n \sec^2 (x^n)]$.

4. (i.) $2x \tan^{-1} x$. (ii.) $\sin^{-1} x$. (iii.) $\frac{1}{2\sqrt{x}(1+x)}$.
(iv.) $\frac{1 - x^2}{1 + 3x^2 + x^4}$. (v.) $\frac{1}{2(1+x^2)}$.

5. $a\omega \sin \omega t$, $a\omega^2 \cos \omega t$.

6. $\dot{x} = 2a\omega \cos^2 \frac{\omega t}{2}$; $\dot{y} = a\omega \sin \omega t$. $\ddot{x} = -a\omega^2 \sin \omega t$; $\ddot{y} = a\omega^2 \cos \omega t$.
The direction of motion at time t makes an angle $\frac{\omega t}{2}$ with the axis of x.

CHAPTER V. (p. 78)

1. (i.) $e^x(1+x)$. (ii.) $x^{m-1}e^{nx}(m+nx)$. (iii.) $(a+bc+cax)e^{cx+d}$.
(iv.) $e^{x \sin^{-1} x}\left(\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}\right)$.

2. (i.) $2xe^{1+x^2}$. (ii.) $2xe^{ax^2}(1+ax^2)$. (iii.) $x^{m-1}e^{ax^n}(m+nax^n)$.
(iv.) $x^{m-1}a^{x^n}(m+nx^n \log a)$.

3. (i.) $x^{m-1}(1+m \log x)$. (ii.) $\frac{1}{(x+1)(x+2)}$. (iii.) $\frac{1}{2\sqrt{x^2-1}}$.
(iv.) $\frac{-6x}{(1-x^2)(4-x^2)}$. (v.) $\frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}}$. (vi.) $\frac{1}{(1-x)\sqrt{x}}$.

4. (i.) $\frac{4x-3}{2\sqrt{(2x+1)(x-2)}}$. (ii.) $\frac{a^2}{(a^2 \pm x^2)^{\frac{3}{2}}}$. (iii.) $\frac{2-5x}{x^3(x-1)^4}$.
(iv.) $x^x(1+\log x)$. (v.) $\frac{mn \cos (m-n)x \sin^{n-1} mx}{\cos^{m+1} nx}$.
(vi.) $\left(\log \frac{1+x}{x} - \frac{1}{x+1}\right)\left(1+\frac{1}{x}\right)^x$.

11. (i.) $\tan \alpha$. (ii.) $\tan n\theta$. (iii.) $-\cot n\theta$. (iv.) $\cot n\theta$. (v.) $-\tan n\theta$.
$r\frac{d\theta}{dr}$ is the tangent of the angle between the radius vector to the point (r, θ), and the tangent to the curve at that point.

13. (i.) $\frac{dy}{dx}=(3x-1)(x-1)$. Max. at $\left(\frac{1}{3}, \frac{4}{27}\right)$.
Min. at (1, 0).

(ii.) $\frac{dy}{dx}=x(5x-2)(x-1)^2$. Max. at origin.
Min. at (·4, ·03).

(iii.) $\frac{dy}{dx}=2(x-1)(x-2)(2x-3)$. Min. at (1, 0); (2, 0).
Max. at $\left(\frac{3}{2}, \frac{1}{16}\right)$.

(iv.) $\frac{dy}{dx}=1-\frac{1}{x^2}$. Max. at (−1, −1).
Min. at (1, 3).

(v.) $\frac{dy}{dx}=2\frac{(x^2-1)}{(x^2+x+1)^2}$. Max. at (−1, 3).
Min. at $\left(1, \frac{1}{3}\right)$.

(vi.) $\frac{dy}{dx}=\frac{4x^2-2x-5}{(x^2+x+1)^2}$. Max. at (−·9, 6·1) nearly.
Min. at (1·4, −·06) nearly.

(vii.) $\frac{dy}{dx}=-\frac{(4x^2-2x-5)}{(x-1)^2(x-2)^2}$. Min. at (−·9, ·16) nearly.
Max. at (1·4, 18·2) nearly.

(viii.) $\frac{dy}{dx}=-\frac{2(x^2-5x+7)}{(x-2)^2(x-4)^2}$. No turning points.

(ix.) $\frac{dy}{dx}=\frac{x^2-8x+10}{(x-4)^2}$. Max. at (1·5, ·1) nearly.
Min. at (6·45, 9·9) nearly.

(x.) $\frac{dy}{dx}=\frac{x^3-2}{x^3}$. Min. at (1·26, 1·89) nearly.

17. (i.) $-\frac{R\theta}{v^2}$. (ii.) $\frac{R}{v}$. (iii.) $\delta p=\frac{R\theta}{v^2}\delta v$.

(iv.) $\delta v=\frac{R\alpha}{p}\delta t$. (v.) $\delta p=-\frac{R(1+\alpha t)}{v^2}\delta v+\frac{\alpha R}{v}\delta t$.

19.
$$\frac{\delta\Delta}{\Delta}=\frac{\delta a}{a}+\frac{\delta b}{b}+\cot C\,\delta C.$$

EXAMPLES ON THE PARABOLA. (p. 83)

	(1)	(2)	(3)
2. Foci, - - -	$\left(-2, -\frac{3}{4}\right)$,	$(-2, -2)$,	$\left(0, -\frac{1}{2}\right)$.
Vertices, - - -	$(-2, -1)$,	$(-2, -3)$,	$\left(-\frac{1}{4}, -\frac{1}{2}\right)$.
Latera recta, - -	$y=-\frac{3}{4}$,	$y=-2$,	$x=0$.
Lengths, - - -	1,	4,	1.
Axes, - - -	$x=-2$,	$x=-2$,	$y=-\frac{1}{2}$.
Tangents at vertices,	$y=-1$,	$y=-3$,	$x=-\frac{1}{4}$.

3. $-5.\ \ 7.$

5. $(1,\ 2),\ \left(\frac{a}{m^2}, \frac{2a}{m}\right).$

6. $x-y-1=0,\quad x+y-3=0.$

7. $x-y+a=0,\quad x+y-3a=0.$
$x+y+a=0,\quad x-y-3a=0.$

EXAMPLES ON THE ELLIPSE. (p. 87)

1. The foci, extremities of the axis, length of latus rectum, and eccentricity are for

(i.) $[\pm 1, 0],\ [\pm 2, 0],\ [9\pm\sqrt{3}],\ 3,\ \frac{1}{2}.$

ii.) $[2, 2],\ [0, 2],\ [3, 2],\ [-1, 2],\ [1, 2+\sqrt{3}],\ [1, 2-\sqrt{3}],\ 3,\ \frac{1}{2}.$

(iii.) $[\pm\sqrt{3}, 1],\ [\pm 2, 1],\ [0, 2],\ [0, 0],\ \frac{1}{2},\ \frac{\sqrt{3}}{2}.$

(iv.) $[0, \pm 1], [0, \pm 2], [\pm\sqrt{3}, 0], 3, \frac{1}{2}.$

EXAMPLES ON THE HYPERBOLA. (p. 90)

1. (i.) $(\pm\sqrt{7}, 0);\ (\pm 2, 0);\ 3;\ \frac{\sqrt{7}}{2}.$

(ii.) $(1\pm\sqrt{7}, 2);\ (3, 2);\ (-1, 2);\ 3;\ \frac{\sqrt{7}}{2}.$

(iii.) $(0,\ -1\pm\sqrt{5});\ (0,\ 0);\ (0,\ -2);\ 8;\ \sqrt{5}.$

(iv.) $(\pm\sqrt{7}, 0);\ (\pm\sqrt{3}, 0);\ \frac{4\sqrt{3}}{3};\ \frac{\sqrt{21}}{3}.$

2. (i.) $(\pm 2\sqrt{2},\ \pm 2\sqrt{2});\ (\pm 2,\ \pm 2).$ (ii.) $(\pm 2\sqrt{2},\ \mp 2\sqrt{2});\ (\pm 2,\ \mp 2).$

CHAPTER VII. (p. 104)

1. (i.) $\frac{(x-a)^4}{4};\ \frac{2\sqrt{ax+b}}{a};\ \frac{2}{3}\sqrt{x}(3+x);\ x-\log(x+3).$

(ii.) $\log\frac{x}{x-1},\ \log\frac{(x-2)^3}{(x-1)},\ \frac{(x-1)^2}{2}+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right),$

$$\frac{x^3}{3}+\frac{x^2}{2}-\frac{1}{2}\log(x^2-x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$$

(iii.) $\sin^{-1}(2x-1);\ \ 2\sqrt{x^2-3x+2}+2\log\left(x-\frac{3}{2}+\sqrt{x^2-3x+2}\right);$

$$\sqrt{(x^2+x+1)}+\frac{1}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right).$$

2. $x\sin^{-1}x+\sqrt{1-x^2};\ \ \frac{x^3}{3}\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log(1+x^2);$

$$\cos 4x\left(\frac{1-8x^2}{32}\right)+\sin 4x\cdot\frac{x}{8};$$

$$\left(\frac{9x^2-2}{27}\right)\sin 3x+\frac{2x}{9}\cos 3x;\ \ \frac{x^{m+1}}{m+1}\log x-\frac{x^{m+1}}{(m+1)^2}-e^{-x}(x^2+2x+2).$$

3. $\frac{1}{6}\log\left(\frac{(1+x)^2}{1-x+x^2}\right)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$.

4. $-\frac{1}{2}\frac{1}{x-1}+\frac{1}{4}\log\left(\frac{x+1}{x-1}\right)$.

5. $\log\frac{(x-3)^2}{x-2}$.

6. (i.) $2(1+x)^{\frac{3}{2}}\left\{\frac{x}{5}-\frac{2}{15}\right\}$. (ii.) $\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$.

8. $\log\tan\frac{\theta}{2}$. $\log\tan\frac{(\theta+a)}{2}$. $\frac{1}{\sqrt{2}}\log\tan\left(\frac{\theta}{2}+\frac{\pi}{8}\right)$.

$\frac{1}{a}\log(a\tan\theta+\sqrt{a^2\tan^2\theta+b^2})$. $\frac{1}{4}\log\frac{2\sec x-1}{2\sec x+1}$.

11. $\int\frac{dx}{5+4\cos x}=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)$. $\int\frac{dx}{5-4\cos x}=\frac{2}{3}\tan^{-1}\left(3\tan\frac{x}{2}\right)$.

$\int\frac{dx}{4+5\cos x}=\frac{1}{3}\log\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}$. $\int\frac{dx}{4-5\cos x}=\frac{1}{3}\log\frac{1+3\tan\frac{x}{2}}{1-3\tan\frac{x}{2}}$.

$$\int\frac{dx}{3+2\sin x}=\frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{1}{5}}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right).$$

$$\int\frac{dx}{3-2\sin x}=\frac{2}{\sqrt{5}}\tan^{-1}\sqrt{5}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right).$$

$$\int\frac{dx}{2+3\sin x}=\frac{2}{\sqrt{5}}\log\frac{\sqrt{5}+\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}{\sqrt{5}-\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}.$$

$$\int\frac{dx}{2-3\sin x}=\frac{2}{\sqrt{5}}\log\frac{1-\sqrt{5}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}{1+\sqrt{5}\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)}.$$

12. $\frac{1}{3}\cos^2\theta\sin\theta+\frac{2}{3}\sin\theta$. $\frac{\cos^3\theta\sin\theta}{4}+\frac{3}{4}\cos\theta\sin\theta+\frac{3}{8}\theta$.

15. $\frac{x^n}{m}\sin mx+\frac{n}{m^2}x^{n-1}\cos mx-\frac{n(n-1)}{m^2}\int x^{n-2}\cos mx\,dx$.

CHAPTER VIII. (p. 126)

1. (i.) 1. (ii.) $\frac{2}{5}\left(e^{-\frac{\pi}{2}}+1\right)$. (iii.) $a^2\log\frac{x_2}{x_1}$.

(iv.) 64. (v.) $3\left(2^{\frac{1}{3}}-\frac{1}{2}\right)$.

2. $\frac{1}{6}$: the difference between the area bounded by the x-axis, the y-axis and the curve, and the area which lies on the negative side of the x-axis.

3. $\frac{343}{12}$.

4. (i.) $\frac{4\pi^3a^2}{3}$. (ii.) $\frac{\pi a^2}{8}, \frac{\pi a^2}{12}, \frac{\pi a^2}{4n}$.

(iii.) $\frac{\pi a^2}{8}, \frac{\pi a^2}{12}, \frac{\pi a^2}{4n}$. (iv.) $a^2 \log \frac{\tan \theta_2}{\tan \theta_1}$.

(v.) $\frac{ab}{2} \tan^{-1} \frac{ab(\tan \theta_2 - \tan \theta_1)}{a^2 \tan \theta_1 \tan \theta_2 + b^2}$. $\frac{ab}{4} \log \frac{(b + a \tan \theta_2)(b - a \tan \theta_1)}{(b + a \tan \theta_1)(b - a \tan \theta_2)}$.

6. (i.) $a \sec \alpha (e^{2\pi \cot \alpha} - 1)$. (ii.) $\frac{a}{2}((2\pi\sqrt{1+4\pi^2}) + \log(2\pi + \sqrt{1+4\pi^2}))$.

(iii.) $\frac{a}{2}(e - e^{-1})$.

7. (i.) $\frac{\pi}{3}$. (ii.) 48π. (iii.) $\frac{5\pi a^3}{3} - \frac{\pi^2 a^3}{2}$.

(iv.) $\frac{3}{5}\pi h^{\frac{5}{3}} a^{\frac{4}{3}}$. (v.) $2ab^2\pi^2$. (vi.) $\frac{4}{3}\pi abc$.

11. (i.) $\frac{2}{3}l$ from that end.

(ii.) On the radius bisecting the arc at a distance $\frac{a \sin \alpha}{\alpha}$ from the centre.

(iii.) $\bar{x} = \frac{4b}{\pi}$, $\bar{y} = \frac{4a}{3\pi}$.

(iv.) On the radius bisecting the sector at a distance $\frac{2a}{3} \frac{\sin \alpha}{\alpha}$ from the centre.

(v.) On the bisector of the chord at a distance $\frac{2}{3} a \frac{\sin^3 \alpha}{\alpha - \sin \alpha \cos \alpha}$ from the centre.

(vi.) The middle point of the radius perpendicular to the base.

12. (i.) $\frac{4}{3}Ml^2$. (rod of length $2l$). (ii.) $\frac{1}{2}Ma^2$. (iii.) $\frac{1}{2}Ma^2$.

(iv.) $\frac{M}{2}(a^2+b^2)$. (v.) $\frac{7}{5}Ma^2$. (vi.) (α) $\frac{Ma^2}{3}$: (β) $M\left(\frac{a^2+b^2}{3}\right)$.

(vii.) (α) $M\frac{b^2}{4}$: (β) $M\frac{a^2}{4}$: (γ) $M\left(\frac{a^2+b^2}{4}\right)$.

(viii.) $M\left(\frac{b^2+c^2}{5}\right)$. (ix.) $M\left(\frac{b^2+c^2}{3}\right)$.

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